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Lectures on n-Dimensional Quasiconformal Mappings

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PREFACE

These notes are based on my lectures at the university of Helsinki in 1967-1968. They were first supposed to be published in another series, and a complete manuscript was given to the publisher in March 1969. When it turned out that the notes could not be published without a considerable delay, they were transferred to the Springer-Verlag in 1971. I have made only some small changes to the original manuscript and added references to the newest literature.

I wish to express my sincere thanks to F. W. Gehring, J. Hesse, R. Näkki, and S. Rickman, who read the manuscript and made valuable suggestions.

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is also defined if A is a subset of a given n-dimensional linear submanifold or of a given n-dimensional sphere in $R^{n'}$, $n' > n$. The subscript n may be omitted if there is no danger of misunderstanding. The measure of a set $A \subset \bar{R}^n$ is defined as the measure of $A \setminus \{\infty\}$.

$\Lambda_\alpha^*(A)$ is the α-dimensional Hausdorff outer measure of A, defined in Section 30. The star is omitted if A is measurable.

The integral of a function $f : A \to \dot{R}^1$ over a set $E \subset A$ is denoted by

$$\int_E f \, dm_n \qquad \text{or} \qquad \int_E f(x) \, dm_n(x) .$$

It is defined if E and f are m_n-measurable and if either f is non-negative or $\int_E |f| \, dm_n < \infty$. In the first case, the integral may have the value ∞. In the second case, f is called integrable over E. The subscript n may again be omitted. Also E can be omitted if $E = R^n$.

The class of Borel sets in a topological space is the smallest σ-algebra which contains the open sets. If A is a Borel set and if T is a topological space, a mapping $f : A \to T$ is said to be a Borel function if $f^{-1}U$ is a Borel set for every open set U in T.

$\Omega_n = m_n(B^n)$ and $\omega_n = m_n(S^n)$. Explicitly,

$$\omega_{n-1} = n\,\Omega_n , \quad \omega_{2k-1} = \frac{2\pi^k}{(k-1)!} , \quad \omega_{2k} = \frac{2^{k+1}\pi^k}{1 \cdot 3 \cdots (2k-1)} .$$

C^k = the class of k times continuously differentiable mappings.

L^p = the class of functions f such that $|f|^p$ is integrable.

If $A : R^n \to R^n$ is a linear mapping, then

$$|A| = \max_{|h|=1} |Ah| , \quad \ell(A) = \min_{|h|=1} |Ah| ,$$

and $\det A$ is the determinant of A.

The words "increasing" and "decreasing" are used in the weak sense. For example, a function $f : (a, b) \to R^1$ is increasing if

INTRODUCTION

By a classical theorem of Liouville, every conformal mapping of a domain in the euclidean n-space R^n, $n \geq 3$, is a restriction of a Möbius transformation, that is, member of the group generated by similarity mappings and inversions in spheres. For this reason, the theory of conformal mappings is essentially 2-dimensional. The situation is different with quasiconformal mappings. Let us consider a diffeomorphism f of a domain $D \subset R^n$ onto a domain $D' \subset R^n$. The derivative of f at a point $x \in D$ is a bijective linear mapping $f'(x) : R^n \to R^n$. The diffeomorphism f is called quasiconformal if the ratio

$$H(x, f) = \frac{\max\limits_{|h|=1} |f'(x)h|}{\min\limits_{|h|=1} |f'(x)h|}$$

is bounded in D. Usually it is more convenient to use a more general definition in which f is not required to be everywhere differentiable. However, it is easy to see that there are plenty of quasiconformal mappings in R^n. For example, if $f : D \to D'$ is a diffeomorphism and if D_0 is a domain whose closure is a compact subset of D, then the restriction $f \mid D_0$ is quasiconformal. Furthermore, while the conformal image of a ball is always a ball or a half space, it is possible to construct a quasiconformal mapping of a ball onto a domain which has non-accessible boundary points (Gehring-Väisälä [2, p. 60]).

2-dimensional quasiconformal mappings were introduced by Grötzsch [1] in 1928. A rather comprehensive treatment of the present state of the theory is given in the excellent books of Ahlfors [3] and Lehto-Virtanen [1]. Higher dimensional quasiconformal mappings were first considered by Soviet mathematicians Lavrentiev [1], Markuševič [1] and Kreines [1] in 1938-1941, but the theory was practically forgotten for 18 years. Since 1959, however, the n-dimensional quasiconformal mappings have been studied rather extensively by a great number of authors in several countries.

The purpose of these notes is to give an exposition of the basic theory of quasiconformal mappings in R^n. The aforementioned books of Ahlfors and Lehto-Virtanen give the historical background, although no previous knowledge is needed on 2-dimensional quasiconformal mappings. In fact, our proofs apply also to the case $n = 2$. However, in this case the proofs could often be simplified, thanks to the Riemann mapping theorem.

We assume that the reader is familiar with the basic facts of the theory of measure and integration. More advanced results of real analysis are given in Chapter 3. Almost all what is needed and much that is included, is contained in the books of Munroe [1] and Saks [1].

We also assume some knowledge on the topology of euclidean spaces. The required facts can be found in the books of Newman [1] and Wilder [1, pp. 51-68]. We shall use the phrase "by Topology" when we are appealing to a topological result (such as the invariance of domain) which is intuitively obvious but often rather profound.

Two important topics have been omitted. We do not prove the theorem of Gehring and Rešetnjak, which states that every 1-quasiconformal mapping is a Möbius transformation for $n \geq 3$. Neither do we present Gehring's theory on the symmetrization of rings. This seems to be unavoidable when deriving sharp bounds in certain modulus estimates. Our results are, therefore, often qualitative rather than quantitative.

The quasiconformal mappings form a subclass of the class of quasiregular mappings, which are not necessarily homeomorphisms. This larger class has not been systematically studied until since

1966, and it is not considered in these notes.

References and brief historical remarks are given at the ends of the sections. The bibliography contains only the publications which are referred to in the text. A very comprehensive bibliography is given in the monograph of Caraman [1].

NOTATION AND TERMINOLOGY

N = the set of positive integers.

Z = the set of integers.

R^1 = the set of real numbers.

R^n = the n-dimensional euclidean space. We identify R^{n-1} with the subspace $x_n = 0$ of R^n.

The letter n denotes always the dimension of the space in question.

$e_1, \ldots, e_n$ = the coordinate unit vectors of R^n. For example, $e_1 = (1, 0, \ldots, 0)$.

The coordinates of a point $x \in R^n$ are denoted by $x_1, \ldots, x_n$. Thus $x = x_1 e_1 + \ldots + x_n e_n$. However, we use subscripts also as indices if there is no danger of misunderstanding. For example, a sequence of points in R^n is often denoted by $x_1, x_2, \ldots$ or by (x_j). The norm of a vector $x \in R^n$ is written as

$$|x| = (x_1^2 + \ldots + x_n^2)^{1/2}.$$

$B^n(x_0, r)$ is the ball $\{x \in R^n \mid |x-x_0| < r\}$. $B^n(r) = B^n(0, r)$. $B^n = B^n(0, 1)$.

$S^{n-1}(x_0, r)$ is the sphere $\{x \in R^n \mid |x-x_0| = r\}$. $S^{n-1}(r) = S^{n-1}(0, r)$. $S^{n-1} = S^{n-1}(0, 1)$. The dimension $n-1$ is sometimes omitted.

$\bar{R}^n = R^n \cup \{\infty\}$ = the one point compactification of R^n. Thus $\bar{R}^n$

is a topological space, homeomorphic to S^n. A metric in $\bar{R}^n$ will be defined in Section 12.

$\dot{R}^1$ = the two-point compactification $R^1 \cup \{-\infty, \infty\}$ of R^1.

Let $A \subset \bar{R}^n$. $\bar{A}$ is the closure of A. ∂A is the boundary of A. $\operatorname{int} A$ is the interior of A. $\underline{C}A$ is the complement of A. All these are taken with respect to $\bar{R}^n$. This justifies the notation $\bar{R}^n$.

By a ball neighborhood of a point $x_0 \in \bar{R}^n$ we mean a ball $B^n(x_0, r)$ if $x_0 \neq \infty$ and a set $\underline{C}B^n(r)$ if $x_0 = \infty$.

The set-theoretical difference of two sets A and B is denoted by $A \setminus B = \{x \mid x \in A,\ x \notin B\}$.

Since points of R^n are treated as vectors, we use the group-theoretic notation $A + B = \{a + b \mid a \in A, b \in B\}$ if A and B are subsets of R^n. Similarly, we define the sets $A - B$, $x + A$, rA, etc., where $x \in R^n$ and $r \in R^1$. $d(A, B)$ is the distance between A and B, and $d(A)$ is the diameter of A.

Let $a, b \in \dot{R}^1$, $a \leq b$. Then $[a, b]$ is the closed interval $\{t \mid a \leq t \leq b\}$. If $a < b$, (a, b) is the open interval $\{t \mid a < t < b\}$. A closed (open) n-interval is the cartesian product of n closed (open) intervals of R^1.

A neighborhood of a point or a set is an open set containing it. A domain is a connected non-empty set.

The notation $f : D \to D'$ includes the assumption that D and D' are domains in $\bar{R}^n$. If Γ is a curve family in D, then Γ' denotes always its image under f.

Let U be an open set in R^n. A mapping $f : U \to R^m$ is differentiable at $x \in U$ if there is a linear mapping $f'(x) : R^n \to R^m$, called the derivative of f at x, such that

$$f(x + h) = f(x) + f'(x)h + |h|\,\varepsilon(x, h)$$

where $\varepsilon(x, h) \to 0$ as $h \to 0$. The jacobian of f at x is denoted by $J(x, f)$.

If $A \subset R^n$, $m_n^*(A)$ is the Lebesgue outer measure of A. $m_n^*(A)$

$a < s < t < b$ implies $f(s) \leq f(t)$.

iff = if and only if.

qc = quasiconformal.

qclly = quasiconformally.

qcty = quasiconformality.

$\Delta\!\!\Delta$ = the end of a proof.

We give a list for other notations, which will be defined in the text and used throughout the rest of the notes.

$\ell(\alpha)$ length of a path 1

$|\alpha|$ locus of a path 1

s_α length function 2

$L(x, f)$ 11

$F(\Gamma)$ 16

$M_p(\Gamma)$, $M(\Gamma)$ modulus 16

$\Gamma_2 > \Gamma_1$ 17

$\Delta(E, F, G)$ 21

$\Delta_0(E, F, G)$ 23

$M_p^S(\Gamma)$ modulus on a manifold 28

b_n constant 28

c_n constant 31

Γ_A path family associated to a ring 33

$R(C_0, C_1) = \underline{C}(C_0 \cup C_1)$ 33

$\varkappa_n(r)$ 34

$q(a, b)$ spherical distance 37

$\lambda_n(r)$ 38

$\lambda_n(r, t)$ 39

$K_I(f)$, $K_O(f)$, $K(f)$ dilatations 41-42

$H_I(A)$, $H_O(A)$, $H(A)$ dilatations 43

$C(f, b)$, $C(f, A)$ cluster sets 52

$\ker_{j\to\infty} E_j$ kernel of a sequence of sets 73

CONTENTS

CHAPTER 1. THE MODULUS OF A CURVE FAMILY

In this chapter we present the theory of moduli of curve families. This concept will be our main tool when studying the properties of qc mappings. In the two-dimensional case it can often be replaced by the conformal mapping technique. The chapter consists of sections 1-12.

1. Paths

1.1. Definitions. A path in $\bar{R}^n$ is a continuous mapping $\alpha : \Delta \to \bar{R}^n$ where Δ is an interval in R^1. The path is said to be closed or open according as Δ is closed or open. The locus $|\alpha|$ of a path $\alpha : \Delta \to \bar{R}^n$ is the point set $\alpha\Delta \subset \bar{R}^n$. A subpath of a path $\alpha : \Delta \to \bar{R}^n$ is a restriction of α to a subinterval of Δ.

Let $\alpha : [a, b] \to R^n$ be a closed path, and let $a = t_0 \leq t_1 \leq \dots \leq t_k = b$ be a subdivision of $[a, b]$. The supremum of the sums

$$\sum_{i=1}^{k} |\alpha(t_i) - \alpha(t_{i-1})|$$

over all subdivisions is called the length of α and denoted by $\ell(\alpha)$. Thus $0 \leq \ell(\alpha) \leq \infty$, and $\ell(\alpha) = 0$ iff α is constant. If $\ell(\alpha) < \infty$, α is rectifiable, otherwise non-rectifiable. Also a path in $\bar{R}^n$ such that $\infty \in |\alpha|$ is non-rectifiable, except for the constant path $\alpha(t) = \infty$, for which we define $\ell(\alpha) = 0$. We remark that the constant paths will never occur when dealing with qc mappings, and they are included in the discussion just for the sake of completeness.

Otherwise stated, α is rectifiable iff it is of bounded variation, and $\ell(\alpha)$ is the total variation of α. Since the total variation is an additive function of an interval, the following result is obvious:

1.2. <u>THEOREM.</u> Let $\alpha : [a, b] \to R^n$ be a rectifiable path, and let $a = t_0 \leq \ldots \leq t_k = b$ be a subdivision of $[a, b]$. Then every restriction $\alpha \mid [t_{i-1}, t_i]$ is rectifiable, and

$$\ell(\alpha) = \sum_{i=1}^{k} \ell(\alpha \mid [t_{i-1}, t_i]) . \quad \Delta$$

Suppose that $\alpha : [a, b] \to R^n$ is a rectifiable path. For each $t \in [a, b]$ we denote $\ell(\alpha \mid [a, t])$ by $s_\alpha(t)$ or only by $s(t)$. The function $s_\alpha : [a, b] \to R^1$ is called the <u>length function</u> of α.

1.3. <u>THEOREM.</u> The length function $s : [a, b] \to R^1$ of a rectifiable path $\alpha : [a, b] \to R^n$ has the following properties:

(1) $a \leq t_1 \leq t_2 \leq b$ implies $\ell(\alpha \mid [t_1, t_2]) = s(t_2) - s(t_1) \geq |\alpha(t_2) - \alpha(t_1)|$.

(2) s is increasing.

(3) s is continuous.

(4) s is absolutely continuous iff α is absolutely continuous.

(5) $s'(t)$ and $\alpha'(t)$ exist a.e. and $s'(t) = |\alpha'(t)|$ a.e.

(6) $\ell(\alpha) \geq \int_a^b s'(t)\, dt = \int_a^b |\alpha'(t)|\, dt$,

where the equality holds iff s (or α) is absolutely continuous.

<u>Proof.</u> (1) follows directly from 1.2 and the definitions. (2) follows from (1). To prove (3), let $t_0 \in [a, b]$. Since s is increasing, it has the left limit h and the right limit h' at t_0. (If

t_o is a or b, only one limit is defined.) We must prove that $h = s(t_o) = h'$. Suppose, for example, that $h < s(t_o)$. Set $r = s(t_o) - h$, and choose $t_1 \in (a, t_o)$. By continuity, there is $q \in (t_1, t_o)$ such that $|\alpha(t) - \alpha(t_o)| < r/3$ for $q \leq t \leq t_o$. Since $\ell(\alpha | [t_1, t_o]) = s(t_o) - s(t_1) \geq r$, there is a subdivision $t_1 = a_o < a_1 < \ldots < a_k = t_o$ such that

$$\sum_{j=1}^{k} |\alpha(a_j) - \alpha(a_{j-1})| > 2r/3 .$$

We may assume that $a_{k-1} > q$. Then $|\alpha(a_k) - \alpha(a_{k-1})| < r/3$ which implies

$$\sum_{j=1}^{k-1} |\alpha(a_j) - \alpha(a_{j-1})| > r/3 .$$

Set $a_{k-1} = t_2$. We then have $\ell(\alpha | [t_1, t_2]) > r/3$. Similarly, we can find $t_3 \in (t_2, t_o)$ such that $\ell(\alpha | [t_2, t_3]) > r/3$. By induction, we obtain a sequence $t_1 < t_2 < \ldots < t_j < \ldots < t_o$ such that $\ell(\alpha | [t_j, t_{j+1}]) > r/3$. By 1.2, we obtain

$$\ell(\alpha | [t_1, t_o]) \geq \ell(\alpha | [t_1, t_p]) = \sum_{j=1}^{p-1} \ell(\alpha | [t_j, t_{j+1}]) > (p-1)r/3$$

for all p. Since α is rectifiable, this leads to a contradiction and proves (3).

To prove (4), we first remark that the "only if" part follows immediately from the inequality in (1). Next assume that α is absolutely continuous. For each $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\sum_{i=1}^{k} |\alpha(b_i) - \alpha(a_i)| < \varepsilon$$

whenever $\Delta_i = [a_i, b_i]$ are non-overlapping subintervals of $[a, b]$, satisfying the condition $\sum m(\Delta_i) < \delta$. Consider such a family $\Delta_1, \ldots, \Delta_k$. Since $s(b_i) - s(a_i) = \ell(\alpha | \Delta_i)$, we can subdivide each Δ_i into intervals $\Delta_{ij} = [a_{ij}, b_{ij}]$ such that $\sum_j |\alpha(b_{ij}) - \alpha(a_{ij})| > s(b_i) - s(a_i) - \varepsilon/k$. Thus

$$\sum_i |s(b_i) - s(a_i)| < \sum_{i,j} |\alpha(b_{ij}) - \alpha(a_{ij})| + \varepsilon < 2\varepsilon ,$$

which proves the absolute continuity of s.

To prove (5), we first remark that since s is increasing and since α is of bounded variation, the derivatives $s'(t)$ and $\alpha'(t)$ exist a.e. Moreover, the inequality of (1) implies that $|\alpha'(t)| \leq s'(t)$ whenever both derivatives exist. Let A be the set of all t such that $s'(t)$ and $\alpha'(t)$ exist and $|\alpha'(t)| < s'(t)$. Let A_k be the set of all $t \in A$ such that

$$\frac{s(q) - s(p)}{q - p} \geq \left| \frac{\alpha(q) - \alpha(p)}{q - p} \right| + \frac{1}{k}$$

whenever $a \leq p \leq t \leq q \leq b$ and $0 < q - p < 1/k$. Since $A = \cup A_k$, it suffices to prove that $m(A_k) = 0$ for a fixed k. Moreover, we may assume that the interval $[a , b]$ is bounded.

Let $\varepsilon > 0$. There is a subdivision $a = t_0 \leq t_1 \leq \cdots \leq t_h = b$ such that $\ell(\alpha) \leq \sum |\alpha(t_j) - \alpha(t_{j-1})| + \varepsilon/k$ and such that $t_j - t_{j-1} < 1/k$ for all $1 \leq j \leq h$. Set $\Delta_j = [t_{j-1} , t_j]$. If $\Delta_j \cap A_k \neq \emptyset$, then $s(t_j) - s(t_{j-1}) \geq |\alpha(t_j) - \alpha(t_{j-1})| + m(\Delta_j)/k$. Hence

$$m(A_k) \leq \sum_{\Delta_j \cap A_k \neq \emptyset} m(\Delta_j) \leq k \sum_{j=1}^{h} (s(t_j) - s(t_{j-1}) - |\alpha(t_j) - \alpha(t_{j-1})|)$$

$$= k\,(\ell(\alpha) - \sum_{j=1}^{h} |\alpha(t_j) - \alpha(t_{j-1})|) \leq \varepsilon .$$

Thus $m(A_k) = 0$, and (5) is proved.

Finally, since $\ell(\alpha) = s(b) = s(b) - s(a)$, (6) is a consequence of (5), (4), and a general theorem in real analysis. $\Delta\!\!\Delta$

2. Change of parameter. Arcs.

2.1. Definition. A path $\alpha : [a , b] \to \bar{R}^n$ is obtained from a path $\beta : [c , d] \to \bar{R}^n$ by an increasing (decreasing) change of parameter if there exists an increasing (decreasing) continuous mapping h of $[a , b]$ onto $[c , d]$ such that $\alpha = \beta \circ h$. If $-\infty < a \leq b < \infty$, the inverse of a path $\alpha : [a, b] \to \bar{R}^n$ is the path $\bar{\alpha} : [a, b] \to \bar{R}^n$, defined by $\bar{\alpha}(t) = \alpha(a + b - t)$.

We omit the easy proof of the following result:

2.2. THEOREM. If α is obtained from β by a change of parameter, then $\ell(\alpha) = \ell(\beta)$. In particular, α is rectifiable iff β is rectifiable. ∆∆

2.3. COROLLARY. $\ell(\bar{\alpha}) = \ell(\alpha)$. ∆∆

2.4. THEOREM. If $\alpha : [a, b] \to R^n$ is a rectifiable path, there exists a unique path $\alpha^0 : [0, c] \to R^n$ with the following properties:

(1) α is obtained from α^0 by an increasing change of parameter.

(2) $\ell(\alpha^0 | [0, t]) = t$ for $0 \le t \le c$. In other words, $s_{\alpha^0}(t) = t$.

Moreover, $c = \ell(\alpha)$, and $\alpha = \alpha^0 \circ s_\alpha$.

Proof. Assume first that α^0 is a path which satisfies the conditions (1) and (2). Then $\alpha = \alpha^0 \circ h$ where $h : [a, b] \to [0, c]$ is increasing. If $a \le t \le b$, 2.2 implies $\ell(\alpha | [0, t]) = \ell(\alpha^0 | [0, h(t)]) = h(t)$. Thus $h = s_\alpha$. This proves the uniqueness of α^0 .

On the other hand, if $s_\alpha(t_1) = s_\alpha(t_2)$, then $\alpha | [t_1, t_2]$ is constant. Hence there exists a well-defined mapping $\alpha^0 : [0, \ell(\alpha)] \to R^n$ such that $\alpha = \alpha^0 \circ s_\alpha$. It is easy to see that α^0 is continuous and satisfies (2). ∆∆

2.5. Definition. The path $\alpha^0 : [0, \ell(\alpha)] \to R^n$ is the normal representation of α . It is also called the parametrization of α by means of its arc length.

2.6. <u>THEOREM.</u> If α is obtained from a rectifiable closed path β by a change of parameter, then $\alpha^0 = \beta^0$ or $\alpha^0 = \overline{\beta^0}$ according as the change is increasing or decreasing.

<u>Proof.</u> Let $\alpha = \beta \circ h$, and assume first that h is increasing. Then $\alpha = \beta^0 \circ s_\beta \circ h$ where $s_\beta \circ h$ is increasing. Since β^0 is a normal representation, the uniqueness part of 2.4 implies $\alpha^0 = \beta^0$. Next assume that h is decreasing. Set $g(t) = \ell(\alpha) - t$. Then $\alpha = \overline{\beta^0} \circ g \circ s_\beta \circ h$ where $g \circ s_\beta \circ h$ is increasing. Since

$$\ell(\overline{\beta^0} | [0\,,\, t]) = \ell(\beta^0 | [\ell(\alpha) - t\,,\, \ell(\alpha)]) = t\,,$$

the uniqueness of α^0 again implies $\alpha^0 = \overline{\beta^0}$. ΔΔ

2.7. <u>Definition.</u> A set $J \subset \bar{R}^n$ is a <u>closed (open) arc</u> if it is homeomorphic to a closed (open) interval $[a\,,\,b]$ $((a\,,\,b))$ where $a < b$.

Suppose that J is a closed arc and that $\alpha : [a\,,\,b] \to J$ and $\beta : [c\,,\,d] \to J$ are homeomorphisms. Then $h = \beta^{-1} \circ \alpha : [a\,,\,b] \to [c\,,\,d]$ is a homeomorphism and hence strictly monotone. Since $\alpha = \beta \circ h$, 2.2 implies $\ell(\alpha) = \ell(\beta)$. Hence the <u>length</u> of J, $\ell(J) = \ell(\alpha)$, is well-defined. If $\ell(J) < \infty$, J is called <u>rectifiable</u>. Theorem 2.6 gives as a corollary:

2.8. <u>THEOREM.</u> Let $\alpha : [a\,,\,b] \to J$ and $\beta : [c\,,\,d] \to J$ be homeomorphisms onto a closed rectifiable arc. Then either $\alpha^0 = \beta^0$ or $\alpha^0 = \overline{\beta^0}$. ΔΔ

One can similarly treat <u>Jordan curves</u>, that is, homeomorphic images of the unit circle. However, they are not needed in these notes. In fact, we will not use arcs either, but they are included here in order that the reader can see the connection between the concepts of these notes and those appearing in the literature.

3. Open paths

3.1. Definition. A path $\alpha : \Delta \to \bar{R}^n$ is locally rectifiable if each closed subpath of α is rectifiable. We then denote $\ell(\alpha) =$ $\sup \ell(\beta)$ over all closed subpaths β of α. If $\ell(\alpha) < \infty$, α is rectifiable.

Clearly, the concepts rectifiable and locally rectifiable coincide for closed paths. Moreover, for a closed path α, the two definitions of $\ell(\alpha)$ are equivalent.

For example, the path $\alpha : (-1, 1) \to R^2$, defined by $\alpha(t) =$ $(t, t \sin(1/t))$, is not locally rectifiable. Its subpath $\alpha | (0, 1)$ is locally rectifiable but not rectifiable. If $|\alpha|$ contains both finite points and ∞, α is not locally rectifiable.

3.2. THEOREM. If $\alpha : (a, b) \to R^n$ is a rectifiable open path, then it has a unique extension to a closed path $\alpha^* : [a, b] \to R^n$. Moreover, $\ell(\alpha) = \ell(\alpha^*)$.

Proof. We must show that the limits of $\alpha(t)$ exist as $t \to a$ and $t \to b$. Suppose that, for example, $\lim_{t \to b} \alpha(t)$ does not exist. We can then find a positive number r and a sequence $t_1 < u_1 < t_2 < u_2 <$ $\ldots < t_j < u_j < \ldots < b$ such that $|\alpha(u_j) - \alpha(t_j)| > r$ for all $j \in N$. Hence

$$\ell(\alpha | [t_1, u_k]) \geq \sum_{j=1}^{k} |\alpha(u_j) - \alpha(t_j)| > kr$$

for every k, which contradicts the rectifiability of α. The last assertion is proved in the same way as the continuity of the length function in 1.3. ΔΔ

3.3. <u>THEOREM.</u> Let $\alpha : (a , b) \to R^n$ be an open path such that α is absolutely continuous on every closed subinterval of (a , b). Then α is locally rectifiable and

$$\ell(\alpha) = \int_a^b |\alpha'(t)|\, dt .$$

<u>Proof.</u> Apply 1.3.(6) to closed subintervals of (a , b). ΔΔ

3.4. <u>Definition.</u> An arc J is <u>locally rectifiable</u> if each closed subarc of J is rectifiable. J is <u>rectifiable</u> if $\ell(J) = \sup \ell(J')$ $< \infty$, where the supremum is taken over all closed subarcs $J' \subset J$.

Alternatively, J is (locally) rectifiable if its homeomorphic parametrization $\alpha : \Delta \to J$ is a (locally) rectifiable path.

<u>4. Line integrals</u>

Throughout this section we assume that $A \subset R^n$ is a Borel set and that $\varrho : A \to \dot{R}^1$ is a non-negative Borel function.

For each rectifiable closed path $\alpha : [a , b] \to A$ we define the <u>line integral</u> of ϱ over α as follows:

$$\int_\alpha \varrho\, ds = \int_0^{\ell(\alpha)} \varrho(\alpha^0(t))\, dt ,$$

where α^0 is the normal representation of α, defined in 2.5. The integral on the right exists, because $\varrho \circ \alpha^0$ is a non-negative Borel function. Its value may be $+\infty$. We also use the notation

$$\int_\alpha \varrho\, ds = \int_\alpha \varrho(x)\, |dx| .$$

4.1. THEOREM. If $\alpha : [a, b] \to A$ is absolutely continuous, then

$$\int_\alpha \varrho \, ds = \int_a^b \varrho(\alpha(t)) \, |\alpha'(t)| \, dt .$$

Proof. Since absolute continuity implies bounded variation, α is rectifiable. Write $\alpha = \alpha^0 \circ s$. Since $s'(t) = |\alpha'(t)|$ a.e. by 1.3, we obtain

$$\int_a^b \varrho(\alpha(t)) \, |\alpha'(t)| \, dt = \int_a^b \varrho(\alpha^0(s(t))) \, s'(t) \, dt .$$

Since s is absolutely continuous by 1.3, the assertion follows from a general theorem in real analysis concerning the change of variable in integrals. ⍋

We also define the line integral of ϱ over a rectifiable closed arc $J \subset A$. If $\alpha : [a, b] \to J$ is a homeomorphism, we set

$$\int_J \varrho \, ds = \int_\alpha \varrho \, ds .$$

It follows from 2.8 that the integral is independent of the choice of α.

If α is a locally rectifiable path such that $|\alpha| \subset A$, we set

$$\int_\alpha \varrho \, ds = \sup \int_\beta \varrho \, ds$$

over all closed subpaths β of α. If α is a rectifiable open path, we also have

$$\int_\alpha \varrho \, ds = \int_{\alpha^*} \varrho \, ds ,$$

where α^* is the closed extension of α, given by 3.2. The integral on the right has an obvious meaning even if ϱ is not defined at the end points. If J is an open locally rectifiable arc in A, we define

$$\int_J \varrho\, ds = \sup \int_{J'} \varrho\, ds$$

over all closed subarcs $J' \subset J$.

4.2. Summary. The line integral $\int_Q \varrho\, ds$ is defined (possibly infinite) if $\varrho : A \to \dot{R}^1$ is a non-negative Borel function and if Q is any of the following quantities:

(1) A rectifiable closed path in A .

(2) A locally rectifiable open path in A .

(3) A rectifiable closed arc in A .

(4) A locally rectifiable open arc in A .

It would also be easy to define the line integral over a rectifiable Jordan curve. Moreover, in the case of arcs and Jordan curves, the definition could also be based on the Hausdorff linear measure without using the parametric representation. In this way the line integral can also be defined for arcs which are not locally rectifiable. (Its value for such arcs would "usually" be ∞).

4.3. Definition. Q is a curve if it is either a path or an arc.

We have thus defined the line integral of a non-negative Borel function over every locally rectifiable curve.

The concept subcurve is clear.

4.4. Remark. In the theory of qc mappings one can make use of either paths or arcs. One reason for this is that the locus of a path is arcwise connected (see Remark 7.11). While most authors have used arcs, we in these notes prefer paths, because they are sometimes technically simpler to deal with. Moreover, arc families cannot be used in the theory of the so-called quasiregular mappings which are not necessarily homeomorphisms. These, however, are not considered in these notes. However, the definition of the modulus and some basic results are formulated for curves.

5. Transformation of line integrals

Suppose that U is an open set in R^n. Let $f: U \to R^m$ be continuous, and let α be a path in U. Then $f \circ \alpha$ is a path in R^m. If it is locally rectifiable and if $\rho: |f \circ \alpha| \to \dot{R}^1$ is a non-negative Borel function, the line integral of ρ over $f \circ \alpha$ is defined. In this section we show how this integral can be estimated by means of a line integral over α. For this purpose we introduce the function

$$L(x, f) = \limsup_{h \to 0} \frac{|f(x+h) - f(x)|}{|h|},$$

defined for $x \in U$. Clearly $0 \le L(x, f) \le \infty$. If f is differentiable at x, then $L(x, f) = |f'(x)|$.

5.1. THEOREM. The function $x \mapsto L(x, f)$ is a Borel function in U.

Proof. Let $a \in R^1$. We must show that $E = \{x \in U \mid L(x, f) < a\}$ is a Borel set. We denote by E_j the set of all $x \in U$ such that $|f(x+h) - f(x)|/|h| \le a - 1/j$ whenever $0 < |h| < 1/j$ and $x+h \in U$. Since f is continuous, every E_j is closed in U. Since $E = \cup E_j$, E is a Borel set. $\Delta\Delta$

5.2. Definition. Let α be a rectifiable closed path in R^n. A mapping $f: |\alpha| \to R^m$ is absolutely continuous on α if $f \circ \alpha^0$ is absolutely continuous on $[0, \ell(\alpha)]$.

For example, suppose that $f : U \to R^m$ is locally lipschitzian, that is, for every compact set $F \subset U$ there is a constant Q_F such that $|f(x) - f(y)| \leq Q_F |x - y|$ for $x, y \in F$. Then f is absolutely continuous on every rectifiable closed path α in U, because $f \circ \alpha^0$ satisfies the Lipschitz condition

$$|f(\alpha^0(u)) - f(\alpha^0(t))| \leq Q |\alpha^0(u) - \alpha^0(t)| \leq Q |u - t| ,$$

where $Q = Q_{|\alpha|}$. In particular, a C^1-mapping $f : U \to R^m$ is absolutely continuous on every rectifiable closed path in U.

5.3. THEOREM. Suppose that U is an open set in R^n and that $f : U \to R^m$ is continuous. Suppose also that $\alpha : \Delta \to U$ is a locally rectifiable path such that f is absolutely continuous on every closed subpath of α. Then $f \circ \alpha$ is locally rectifiable. If $\varrho : |f \circ \alpha| \to \dot{R}^1$ is a non-negative Borel function, then

$$\int_{f \circ \alpha} \varrho \, ds \leq \int_{\alpha} \varrho(f(x)) \, L(x, f) \, |dx| .$$

Proof. We first assume that Δ is a closed interval $[a, b]$. Since $f \circ \alpha^0$ is absolutely continuous, it is rectifiable. By 2.2, the path $f \circ \alpha = f \circ \alpha^0 \circ s_\alpha$ is rectifiable. By 5.1, both integrals are defined. Set $\ell(\alpha) = p$, $\ell(f \circ \alpha) = \ell(f \circ \alpha^0) = q$, and let $s : [0, p] \to [0, q]$ be the length function of $f \circ \alpha^0$. Then s is absolutely continuous by 1.3. Setting $\beta = (f \circ \alpha)^0$, we obtain by 2.6 $\beta \circ s = (f \circ \alpha^0)^0 \circ s = f \circ \alpha^0$. Using the transformation formula for Lebesgue integrals we obtain

$$\int_{f \circ \alpha} \varrho \, ds = \int_0^q \varrho(\beta(t)) \, dt = \int_0^p \varrho(\beta(s(u))) \, s'(u) \, du = \int_0^p \varrho(f(\alpha^0(u))) \, s'(u) \, du .$$

By 1.3, $s'(u) = |(f \circ \alpha^0)'(u)|$ a.e. Consider u such that $(f \circ \alpha^0)'(u)$ and $\alpha^{0\prime}(u)$ exist. Since α^0 is a normal representation, we can find a sequence (r_j) such that $r_j \to 0$ and

$\alpha^0(u+r_j) \neq \alpha^0(u)$. We then have

$$|(f \circ \alpha^0)'(u)| = \lim_{j\to\infty} \frac{|f(\alpha^0(u+r_j)) - f(\alpha^0(u))|}{|\alpha^0(u+r_j) - \alpha^0(u)|} \frac{|\alpha^0(u+r_j) - \alpha^0(u)|}{|r_j|}$$

$$\leq L(\alpha^0(u) , f) \, |\alpha^{0\prime}(u)| .$$

On the other hand, 1.3 implies that $|\alpha^{0\prime}(u)| = 1$ a.e. Combining the above results we obtain $s'(u) \leq L(\alpha^0(u) , f)$ a.e. Hence

$$\int_{f\circ\alpha} \varrho \, ds \leq \int_0^p \varrho(f(\alpha^0(u))) \, L(\alpha^0(u) , f) \, du = \int_\alpha \varrho(f(x)) \, L(x , f) \, |dx| .$$

The theorem is thus proved in the case where α is a closed path. If α is open, the inequality holds for every closed subpath of α and hence for the whole α . ∆∆

5.4. COROLLARY. If $f : U \to R^m$ is a C^1-mapping and if $\alpha : \Delta \to U$ is locally rectifiable, then $f \circ \alpha$ is locally rectifiable. If $\varrho : |f \circ \alpha| \to \dot{R}^1$ is a non-negative Borel function, then

$$\int_{f\circ\alpha} \varrho \, ds \leq \int_\alpha \varrho(f(x)) \, |f'(x)| \, |dx| . \quad ∆∆$$

5.5. Definition. Let D and D' be domains in R^n . A homeomorphism $f : D \to D'$ is conformal if $f \in C^1$ and if $|f'(x)h| = |f'(x)||h|$ for every $x \in D$ and $h \in R^n$. If D and D' are domains in $\bar{R}^n$, a homeomorphism $f : D \to D'$ is conformal if its restriction to $D \setminus \{\infty , f^{-1}(\infty)\}$ is conformal.

Alternatively, a C^1-homeomorphism f is conformal iff $|f'(x)|^n = |J(x , f)|$ for all $x \in D$.

It is well known that a 2-dimensional diffeomorphism with positive jacobian is conformal iff it is complex analytic. As mentioned in the introduction, for $n \geq 3$ every conformal mapping is a Möbius transformation. By a Möbius transformation we mean a mapping $f : \bar{R}^n \to \bar{R}^n$ which is composed of a finite number of the following elementary transformations:

(1) Translation: $f(x) = x + a$.

(2) Stretching: $f(x) = rx$, $r > 0$.

(3) Orthogonal mapping: f is linear and $|f(x)| = |x|$ for all $x \in R^n$.

(4) Inversion in a sphere $S(a, r)$: $f(x) = a + \dfrac{r^2(x-a)}{|x-a|^2}$.

In fact, every Möbius transformation can be expressed as a composite mapping of inversions. A Möbius transformation can always be written in one of the following forms: $f(x) = rTx + a$ or $f(x) = I(rTx + a)$ where $r > 0$, $a \in R^n$, T is an orthogonal mapping and I is an inversion. The first case occurs iff $f(\infty) = \infty$, and f is then called a similarity mapping. It is easy to see that every elementary transformation, and hence every Möbius transformation, is conformal. For the converse, see Remark 5.8.

The following result is proved exactly as Theorem 5.3:

5.6. THEOREM. If, in the situation of Theorem 5.3, the limit

$$\lim_{h \to 0} \frac{|f(x+h) - f(x)|}{|h|}$$

exists for every $x \in |\alpha|$, then

$$\int_{f\circ\alpha} \rho \, ds = \int_{\alpha} \rho(f(x)) \, L(x, f) \, |dx| .$$

In particular, if f is a conformal mapping, then

$$\int_{f\circ\alpha} \rho \, ds = \int_{\alpha} \rho(f(x)) \, |f'(x)| \, |dx| . \quad \Delta$$

As an application of Theorem 5.3 we prove the following intuitively obvious result:

5.7. <u>THEOREM.</u> Let ϱ be a non-negative Borel function, defined on an interval $[p, q]$, and let $\alpha : [a, b] \to R^n$ be a rectifiable path such that $p \leq |\alpha(t)| \leq q$ for all $a \leq t \leq b$. Then

$$\int_\alpha \varrho(|x|)\,|dx| \geq \left| \int_{|\alpha(a)|}^{|\alpha(b)|} \varrho(u)\,du \right| .$$

<u>Proof.</u> Define $f : R^n \to R^1$ by $f(x) = |x|$. Then $L(x, f) = 1$ for every $x \in R^n$. Since $|f(x) - f(y)| \leq |x - y|$, f is absolutely continuous on α. Applying 5.3 we obtain

$$\int_{f \circ \alpha} \varrho\,ds \leq \int_\alpha \varrho(f(x))\,|dx| = \int_\alpha \varrho(|x|)\,|dx| .$$

Set $\beta = (f \circ \alpha)^0$ and $c = \ell(f \circ \alpha)$. Then $\beta(0) = |\alpha(a)|$ and $\beta(c) = |\alpha(b)|$. We may assume that $|\alpha(a)| \leq |\alpha(b)|$. By 1.3, $|\beta'(t)| = 1$ a.e. In order to use the formula for the change of variable, we set $\varrho_k(u) = \min(\varrho(u), k)$. Since β is absolutely continuous, this formula (Graves [1, p. 221]) gives

$$\int_{|\alpha(a)|}^{|\alpha(b)|} \varrho_k(u)\,du = \int_0^c \varrho_k(\beta(t))\,\beta'(t)\,dt \leq \int_0^c \varrho(\beta(t))\,dt = \int_{f \circ \alpha} \varrho\,ds .$$

Combining the above inequalities and letting $k \to \infty$ yields the theorem. ΔΔ

5.8. <u>Remark.</u> Liouville [1] proved in 1850 that if $n \geq 3$ and if $f : D \to D'$ is conformal and C^3, then f is a restriction of a Möbius transformation. It has been surprisingly difficult to weaken the differentiability hypotheses. For C^1-mappings the result was proved by Hartman [1] in 1959. For a still more general result, see Remark 13.7.2. A simple proof for C^4-mappings has been given by Nevanlinna [1].

6. The modulus

6.1. We are now ready to define the modulus of a curve family. Suppose that Γ is a curve family in $\bar{R}^n$. That is, the elements of Γ are curves in $\bar{R}^n$. We denote by $F(\Gamma)$ the set of all non-negative Borel functions $\varrho : R^n \to \dot{R}^1$ such that

$$\int_\gamma \varrho \, ds \geq 1$$

for every locally rectifiable curve $\gamma \in \Gamma$. For each $p \geq 1$ we set

$$M_p(\Gamma) = \inf_{\varrho \in F(\Gamma)} \int_{R^n} \varrho^p \, dm .$$

If $F(\Gamma) = \emptyset$, we define $M_p(\Gamma) = \infty$. This happens only if Γ contains a constant path (which will never occur in these notes), because otherwise the constant function $\varrho(x) = \infty$ belongs to $F(\Gamma)$. Clearly $0 \leq M_p(\Gamma) \leq \infty$.

The number $M_p(\Gamma)$ is called the <u>p-modulus</u> of Γ. The most important for our purposes is the case $p = n$. We shall denote $M_n(\Gamma)$ simply by $M(\Gamma)$ and call it the <u>modulus</u> of Γ.

In the literature, one often uses the <u>extremal length</u> of Γ. This is simply equal to $1/M(\Gamma)$. The modulus is perhaps a more natural concept, for it has the following measure-theoretic property:

6.2. <u>THEOREM.</u> M_p is an outer measure in the space of all curves in $\bar{R}^n$. That is,

(1) $M_p(\emptyset) = 0$,

(2) $\Gamma_1 \subset \Gamma_2$ implies $M_p(\Gamma_1) \leq M_p(\Gamma_2)$.

(3) $M_p(\bigcup_{i=1}^{\infty} \Gamma_i) \leq \sum_{i=1}^{\infty} M_p(\Gamma_i)$.

<u>Proof.</u> Since the zero function belongs to $F(\emptyset)$, $M_p(\emptyset) = 0$. If

$\Gamma_1 \subset \Gamma_2$, then $F(\Gamma_1) \supset F(\Gamma_2)$, whence $M_p(\Gamma_1) \leq M_p(\Gamma_2)$. To prove (3), we may assume that every $M_p(\Gamma_i) < \infty$. For $\varepsilon > 0$ pick $\varrho_i \in F(\Gamma_i)$ such that

$$\int \varrho_i^p \, dm < M_p(\Gamma_i) + \varepsilon/2^i .$$

Then the function $\varrho = (\sum \varrho_i^p)^{1/p}$ belongs to $F(\Gamma)$, since $\varrho \geq \varrho_i$ for all $i \in N$. Thus

$$M_p(\Gamma) \leq \int \varrho^p \, dm = \sum \int \varrho_i^p \, dm < \varepsilon + \sum M_p(\Gamma_i) .$$

Letting $\varepsilon \to 0$ yields (3). ∆

In view of this theorem, it is natural to use the phrase "almost every curve" to mean "every curve except for a family of modulus zero".

6.3. Definition. Let Γ_1 and Γ_2 be curve families in $\bar{R}^n$. We say that Γ_2 is minorized by Γ_1 and denote $\Gamma_2 > \Gamma_1$ if every $\gamma \in \Gamma_2$ has a subcurve which belongs to Γ_1.

6.4. THEOREM. If $\Gamma_1 < \Gamma_2$, then $M_p(\Gamma_1) \geq M_p(\Gamma_2)$.

Proof. Obviously $F(\Gamma_1) \subset F(\Gamma_2)$. ∆

6.5. Remark. If $\Gamma_1 \supset \Gamma_2$, then $\Gamma_1 < \Gamma_2$. Thus 6.2.(2) is a special case of 6.4. In fact, both results are special cases of Theorem 6.7 below. Roughly speaking, $M_p(\Gamma)$ is large if there are many curves in Γ or if the curves of Γ are short.

6.6. Definition. The curve families $\Gamma_1, \Gamma_2, \ldots$ are called separate if there exist disjoint Borel sets E_i in R^n such that if $\gamma \in \Gamma_i$ is locally rectifiable, then $\int_\gamma g_i \, ds = 0$ where g_i is the characteristic function of $\underline{C}E_i$.

6.7. THEOREM. If $\Gamma_1, \Gamma_2, \ldots$ are separate and if $\Gamma < \Gamma_i$ for all i, then

$$M_p(\Gamma) \geq \sum M_p(\Gamma_i) .$$

Proof. For $\rho \in F(\Gamma)$ set $\rho_i(x) = (1 - g_i(x))\,\rho(x)$. Then $\rho_i \in F(\Gamma_i)$, whence

$$\sum M_p(\Gamma_i) \leq \sum \int \rho_i^p \, dm = \sum \int_{E_i} \rho^p \, dm \leq \int \rho^p \, dm .$$

Thus $\sum M_p(\Gamma_i) \leq M_p(\Gamma)$. ΔΔ

6.8. Remark. The above definitions and results apply also to the case $n = 1$. However, from now on we shall always assume that $n \geq 2$ unless otherwise stated. For example, Theorem 6.9 below is not true in R^1.

It is clear from the definition of the modulus that the curves which are not locally rectifiable play no role in the modulus. Hence $M_p(\Gamma) = M_p(\Gamma_0)$ where Γ_0 is the family of all locally rectifiable curves in Γ. In other words, p-almost every curve in $\bar{R}^n$ is locally rectifiable for every $p \geq 1$. We next show that in the important case $p = n$, one can even restrict oneself to rectifiable curves.

If Γ is a curve family in $\bar{R}^n$, we denote by $F_r(\Gamma)$ the family of all non-negative Borel functions $\rho : R^n \to \dot{R}^1$ such that $\int_\gamma \rho \, ds \geq 1$ for every rectifiable $\gamma \in \Gamma$. If the elements of Γ are closed paths or closed arcs, then $F_r(\Gamma) = F(\Gamma)$. In general, $F(\Gamma) \subset F_r(\Gamma)$.

6.9. THEOREM. If Γ is a curve family in $\bar{R}^n$,

$$M(\Gamma) = \inf_{\rho \in F_r(\Gamma)} \int \rho^n \, dm .$$

Proof. Denote by q the right-hand side of the equality. Since $F(\Gamma) \subset F_r(\Gamma)$, $M(\Gamma) \geq q$. To prove the reverse inequality, we define $\rho_0 : R^n \to \dot{R}^1$ by $\rho_0(x) = 1/|x| \log |x|$ for $|x| \geq 2$ and $\rho_0(x) = 1$ for $|x| < 2$. By direct computation we obtain

$$\int \rho_0^n \, dm = 2^n \Omega_n + \omega_{n-1}/(n-1)(\log 2)^{n-1} < \infty .$$

We next show that the line integral of ρ_0 is ∞ over every locally rectifiable curve γ which is not rectifiable. If γ is bounded, then $\rho_0(x) \geq a > 0$ on $|\gamma|$, and the assertion is clear. If γ is unbounded, we choose a point x on $|\gamma|$ such that $|x| \geq 2$. From 5.7 it follows that

$$\int_\gamma \rho_0 \, ds \geq \int_{|x|}^{\infty} \frac{dr}{r \log r} = \infty .$$

Now let $\rho \in F_r(\Gamma)$. For $\varepsilon > 0$ we set $\rho_\varepsilon = (\rho^n + \varepsilon^n \rho_0^n)^{1/n}$ and show that $\rho_\varepsilon \in F(\Gamma)$. Since $\rho_\varepsilon > \rho$,

$$\int_\gamma \rho_\varepsilon \, ds \geq \int_\gamma \rho \, ds \geq 1$$

for every rectifiable $\gamma \in \Gamma$. If γ is non-rectifiable, then

$$\int_\gamma \rho_\varepsilon \, ds \geq \varepsilon \int_\gamma \rho_0 \, ds = \infty .$$

Hence $\rho_\varepsilon \in F(\Gamma)$, and we obtain

$$M(\Gamma) \leq \int \rho_\varepsilon^n \, dm = \int \rho^n \, dm + \varepsilon^n \int \rho_0^n \, dm .$$

Since $\varepsilon > 0$ and $\rho \in F_r(\Gamma)$ are arbitrary, this proves $M(\Gamma) \leq q$. Δ

6.10. COROLLARY. If Γ_r is the family of all rectifiable curves in Γ, then $M(\Gamma_r) = M(\Gamma)$. Δ

6.11. COROLLARY. The family of all non-rectifiable curves in $\bar{R}^n$ has modulus zero. In other words, almost every curve in $\bar{R}^n$ is rectifiable. ∆

6.12. Remarks. The concept of the modulus of a curve family was first published by Ahlfors and Beurling [1] in 1951. The notion was generalized in 1957 by Fuglede [1] who also gave the measure-theoretic interpretation 6.2. One can similarly consider families of higher-dimensional surfaces, but they are not used in these notes.

Some authors define $M_p(\Gamma)$ using only continuous functions $\rho \in F(\Gamma)$, but this leads to a number $M_p^*(\Gamma)$ which is in general greater than $M_p(\Gamma)$. However, $M_p^*(\Gamma) = M_p(\Gamma)$ in the most important cases, for example if $\Gamma = \Gamma_A$ is the path family associated to a ring A (see Section 11). On the other hand, one obtains $M_p(\Gamma)$ by using only lower semicontinuous functions ρ (Fuglede [1, p. 173]).

7. Examples

Given a curve family Γ, it is usually a very difficult task to compute $M_p(\Gamma)$. However, it is often easy to find an upper bound for $M_p(\Gamma)$, for if we take any $\rho \in F(\Gamma)$, then $M_p(\Gamma) \leq \int \rho^p \, dm$. As an example, we prove the following inequality:

7.1. THEOREM. Suppose that the curves of a family Γ lie in a Borel set $G \subset \bar{R}^n$ and that $\ell(\gamma) \geq r > 0$ for every locally rectifiable $\gamma \in \Gamma$. Then

$$M_p(\Gamma) \leq \frac{m(G)}{r^p} .$$

Proof. Define $\rho : R^n \to R^1$ by $\rho(x) = 1/r$ for $x \in G$ and $\rho(x) = 0$ otherwise. Then $\rho \in F(\Gamma)$, and the inequality follows. ∆

It is usually much more difficult to find a significant (i.e. positive) lower bound for $M_p(\Gamma)$. To do so, we must consider an arbitrary $\varrho \in F(\Gamma)$ and prove that the integral of ϱ^p is $\geq$ a fixed number. This is usually done by a method which involves Hölder's inequality and Fubini's theorem. We illustrate this by actually computing the moduli of certain important curve families.

We first introduce a notation which will be used throughout these notes. If E, F, G are subsets of $\bar{R}^n$, we let $\Delta(E, F, G)$ denote the family of all closed paths which join E and F in G. More precisely, a path $\gamma : [a, b] \to \bar{R}^n$ belongs to $\Delta(E, F, G)$ iff (1) one of the end points $\gamma(a), \gamma(b)$ belongs to E and the other to F, and (2) $\gamma(t) \in G$ for $a < t < b$.

7.2. <u>The cylinder.</u> Let E be a Borel set in R^{n-1} and let $h > 0$. Set

$$G = \{x \in R^n \mid (x_1, \ldots, x_{n-1}) \in E \text{ and } 0 < x_n < h\}.$$

Then G is a cylinder with bases E and $F = E + he_n$ and with height h. Set $\Gamma = \Delta(E, F, G)$. We show that

$$M_p(\Gamma) = \frac{m_{n-1}(E)}{h^{p-1}} = \frac{m(G)}{h^p}.$$

Since $\ell(\gamma) \geq h$ for every $\gamma \in \Gamma$, 7.1 implies $M_p(\Gamma) \leq m(G)/h^p$. Let ϱ be an arbitrary function in $F(\Gamma)$. For each $y \in E$ let $\gamma_y : [0, h] \to R^n$ be the vertical segment $\gamma_y(t) = y + te_n$. Then $\gamma_y \in \Gamma$. Assuming that $p > 1$ we obtain by Hölder's inequality

$$1 \leq \Big(\int_{\gamma_y} \varrho \, ds\Big)^p \leq h^{p-1} \int_0^h \varrho(y+te_n)^p \, dt .$$

Integration over $y \in E$ yields by Fubini's theorem

$$m_{n-1}(E) \leq h^{p-1} \int_E dm_{n-1} \int_0^h \varrho(y+te_n)^p \, dt = h^{p-1} \int_G \varrho^p \, dm \leq h^{p-1} \int \varrho^p \, dm .$$

Since this holds for every $\varrho \in F(\Gamma)$, we obtain $M_p(\Gamma) \geq$

$m_{n-1}(E)/h^{p-1}$.

The proof for $p=1$ is somewhat simpler. ΔΔ

7.3. Remark. The above proof also shows that $M_p(\Gamma) = M_p(\Gamma_0)$ where Γ_0 is the subfamily of Γ consisting of the vertical segments γ_y .

7.4. Remark. In Example 7.2, $M_p(\Gamma)$ is invariant under similarity mappings iff $p = n$. This is the reason why the case $p = n$ is so important in the theory of qc mappings. Indeed, we shall show in the next section that $M(\Gamma)$ is a conformal invariant.

7.5. The spherical ring. If $0 < a < b < \infty$, the domain $A = B^n(b) \setminus \bar{B}^n(a)$ is called a spherical ring. Let $E = S(a)$, $F = S(b)$ and $\Gamma_A = \Delta(E, F, A)$. We shall prove that

$$M(\Gamma_A) = \omega_{n-1} (\log \frac{b}{a})^{1-n} .$$

Let $\varrho \in F(\Gamma_A)$. For each unit vector $y \in S^{n-1}$ we let $\gamma_y : [a, b] \to R^n$ be the radial segment, defined by $\gamma_y(t) = ty$. By Hölder's inequality we obtain

$$1 \leq \Big(\int_{\gamma_y} \varrho \, ds\Big)^n \leq \int_a^b \varrho(ty)^n t^{n-1} \, dt \Big(\int_a^b t^{-1} \, dt\Big)^{n-1}$$

$$= (\log \frac{b}{a})^{n-1} \int_a^b \varrho(ty)^n t^{n-1} \, dt .$$

Integrating over $y \in S^{n-1}$ yields

$$(7.6) \qquad \omega_{n-1} \leq (\log \frac{b}{a})^{n-1} \int \varrho^n \, dm .$$

Taking the infimum over all $\varrho \in F(\Gamma)$ we obtain

$$\omega_{n-1} \leq (\log \frac{b}{a})^{n-1} M(\Gamma_A) .$$

On the other hand, we have equality in (7.6) if we define $\varrho(x) =$

$1/|x| \log(b/a)$ for $x \in A$ and $\varrho(x) = 0$ otherwise. It is easy to see by 5.7 that this ϱ belongs to $F(\Gamma)$.

7.7. Remark. Let Y be a Borel set in S^{n-1} and let C be the cone $\{x \in R^n \mid x/|x| \in Y\}$. Set $\Gamma = \{\gamma \in \Gamma_A \mid |\gamma| \subset C\}$ where A is as above. Then the method of 7.5 yields

$$M(\Gamma) = m_{n-1}(Y)\,(\log\frac{b}{a})^{1-n}.$$

In fact, $M(\Gamma) = M(\Gamma_0)$ where Γ_0 is the family of all radial segments γ_y, $y \in Y$.

7.8. The degenerate ring. Let $\Gamma = \Delta(E, F, G)$ where $E = \{0\}$, $F = S^{n-1}(b)$ and $G = B^n(b) \setminus \{0\}$. Since $\Gamma > \Gamma_A$ for every spherical ring $A = B^n(b) \setminus \bar{B}^n(a)$, we obtain from 6.4 and 7.5

$$M(\Gamma) \leq M(\Gamma_A) \leq \omega_{n-1}\,(\log\frac{b}{a})^{1-n}.$$

Since this holds for every $a > 0$, $M(\Gamma) = 0$.

7.9. Paths through a point. Let $x_0 \in \bar{R}^n$ and let Γ be the family of all non-constant paths γ such that $x_0 \in |\gamma|$. We show that $M(\Gamma) = 0$. If $x_0 = \infty$, this is trivial. If $x_0 \neq \infty$, we let Γ_k be the family of all $\gamma \in \Gamma$ such that $|\gamma|$ meets $S(x_0, 1/k)$. Then $M(\Gamma_k) = 0$ by 7.8 and 6.4. On the other hand, $\Gamma = \cup \Gamma_k$, whence by 6.2, $M(\Gamma) \leq \sum M(\Gamma_k) = 0$. Hence, given $x_0 \in \bar{R}^n$, almost every non-constant path omits x_0.

We next consider open paths and introduce the following notation: Given three sets E, F, G in $\bar{R}^n$, we let $\Delta_0(E, F, G)$ be the family of all open paths γ joining E and F in G in the following sense: $|\gamma| \subset G$ and $\overline{|\gamma|} \cap E \neq \emptyset \neq \overline{|\gamma|} \cap F$.

7.10. THEOREM. $M(\Delta_0(E, F, G)) = M(\Delta(E, F, G))$.

Proof. Set $\Gamma = \Delta(E, F, G)$ and $\Gamma_0 = \Delta_0(E, F, G)$. Since $\Gamma_0 < \Gamma$, $M(\Gamma_0) \geq M(\Gamma)$. To prove the reverse inequality, it suffices to show, in view of 6.9, that $F(\Gamma) \subset F_r(\Gamma_0)$. Suppose that $\varrho \in F(\Gamma)$ and that γ is a rectifiable path in Γ_0. Let γ^* be the closed extension of γ, given in 3.2. Then $|\gamma^*| = \overline{|\gamma|}$ meets both E and F. Choose t_1, t_2 such that $\gamma^*(t_1) \in E$, $\gamma^*(t_2) \in F$, and assume that $t_1 \leq t_2$. Then $\beta = \gamma^* | [t_1, t_2] \in \Gamma$, whence

$$\int_\gamma \varrho \, ds = \int_{\gamma^*} \varrho \, ds \geq \int_\beta \varrho \, ds \geq 1 .$$

Consequently, $\varrho \in F_r(\Gamma_0)$. $\Delta\!\!\Delta$

7.11. Remark. One can also consider the family $\Delta_a(E, F, G)$ of all closed arcs which join E and F in G. This family has the same modulus as $\Delta(E, F, G)$. To prove this, choose a parametric representation for each arc in $\Delta_a(E, F, G)$. We then obtain a subfamily Γ of $\Delta(E, F, G)$. Thus $M(\Delta_a) = M(\Gamma) \leq M(\Delta)$. On the other hand, the locus of every path is arcwise connected by Topology. Hence for each $\gamma \in \Delta$ there is an arc $J \in \Delta_a$ such that $J \subset |\gamma|$. This implies $F(\Delta_a) \subset F(\Delta)$ and thus $M(\Delta_a) \geq M(\Delta)$.

Also the family of all open arcs joining E and F in G leads to the same modulus.

7.12. Remark. The reader can now go through sections 13-16 and learn the basic facts about qc mappings. However, for a deeper study we need estimates for the moduli of several types of path families. These will be given in sections 8-12.

8. Moduli in a conformal mapping

Suppose that A is a subset of $\bar{R}^n$ and that $f : A \to \bar{R}^n$ is continuous. If Γ is a family of paths in A, then the family $\Gamma' = \{f \circ \gamma \mid \gamma \in \Gamma\}$ is called the image of Γ under f.

8.1. THEOREM. If $f : D \to D'$ is conformal (see 5.5), then $M(\Gamma') = M(\Gamma)$ for every path family Γ in D.

Proof. By 7.9, we may assume that the paths of Γ and Γ' do not go through ∞. Let $\varrho' \in F(\Gamma')$. Define $\varrho(x) = \varrho'(f(x))|f'(x)|$ for $x \in D$ and $\varrho(x) = 0$ for $x \in \complement D$. By 5.6,

$$\int_\gamma \varrho \, ds = \int_{f\circ\gamma} \varrho' \, ds \geq 1$$

for every locally rectifiable $\gamma \in \Gamma$. Hence $\varrho \in F(\Gamma)$, which implies

$$M(\Gamma) \leq \int \varrho^n \, dm = \int_D \varrho'(f(x))^n |J(x, f)| \, dm(x) = \int_{D'} \varrho'^n \, dm \leq \int \varrho'^n \, dm .$$

Since this is true for every $\varrho' \in F(\Gamma')$, $M(\Gamma) \leq M(\Gamma')$. Since f^{-1} is conformal, the reverse inequality is also true. ∆

We found in 7.4 that $M_p(\Gamma)$ is not a conformal invariant if $p \neq n$. We give a formula which shows what happens to $M_p(\Gamma)$ in a conformal linear mapping. Let $k > 0$ and define $f : R^n \to R^n$ by $f(x) = kx$. The image of a path family Γ under f is denoted by $k\Gamma$.

8.2. THEOREM. $M_p(k\Gamma) = k^{n-p} M_p(\Gamma)$.

Proof. Observing that $|f'(x)| = k$, the proof can be carried out as in 8.1. ∆

9. Symmetrization of real functions

In this section we prove an inequality which will be needed in 10.2. Suppose that $f : R^1 \to R^1$ is a non-negative bounded measurable function. For $y > 0$ we set $M(y) = m(\{x \mid f(x) \geq y\})$. Next we define the function $f^* : R^1 \to \dot{R}^1$ by

$$f^*(x) = \inf \{y \mid M(y) \leq 2x\} \quad \text{for} \quad x \geq 0 ,$$
$$f^*(x) = f^*(-x) \quad \text{for} \quad x < 0 .$$

The function $M : (0, \infty) \to \dot{R}^1$ is clearly decreasing, and $M(y) = 0$ for large y. Hence $f^*(x)$ is finite and decreasing for $x \geq 0$. The function f^* is called the <u>symmetrization</u> of f.

It follows from the definitions that $f \leq g$ implies $f^* \leq g^*$. Moreover, if $f(x)$ is strictly decreasing and continuous for $x \geq 0$ and if $f(-x) = f(x)$, then $f^* = f$.

For example, if f is the characteristic function of a measurable set E, $M(y) = m(E)$ for $0 < y \leq 1$ and $M(y) = 0$ for $y > 1$. Hence f^* is the characteristic function of the interval $(-m(E)/2 , m(E)/2)$. Suppose, more generally, that f is a simple function. This means that $f = a_1u_1 + \dots + a_ku_k$ where the functions u_i are characteristic functions of disjoint measurable sets A_i and $0 < a_1 < a_2 < \dots < a_k$. Set $\alpha_i = a_i - a_{i-1}$ with $\alpha_1 = a_1$, and denote by f_i the characteristic function of $B_i = A_i \cup \dots \cup A_k$. Then

$$f = \sum_{i=1}^{k} \alpha_i f_i . \tag{9.1}$$

It follows from the definitions that f^* is a simple function, defined by $f^*(x) = a_i$ for $m(B_{i+1})/2 \leq |x| < m(B_i)/2$, $f^*(x) = a_k$ for $|x| < m(B_k)/2$, and $f^*(x) = 0$ for $|x| \geq m(B_1)/2$. Hence

$$f^* = \sum_{i=1}^{k} \alpha_i f_i^* .$$

9.2. THEOREM. Let f and g be non-negative bounded measurable functions in R^1. Then

$$\int_{R^1} fg\,dm \le \int_{R^1} f^* g^*\,dm .$$

Proof. Suppose first that f and g are characteristic functions of sets E and F, respectively. Assuming that $m(E) \le m(F)$ we obtain

$$\int fg\,dm = m(E \cap F) \le m(E) = \int f^* g^*\,dm .$$

Suppose next that f and g are simple functions. Let $f = \sum \alpha_i f_i$ and $g = \sum \beta_j g_j$ be their representations in the form (9.1). Then

$$\int fg\,dm = \sum \alpha_i \beta_j \int f_i g_j\,dm \le \sum \alpha_i \beta_j \int f_i^* g_j^*\,dm = \int f^* g^*\,dm .$$

Consider now the general case. Choose increasing sequences of simple functions (f_j) and (g_j) such that $f_j \to f$ and $g_j \to g$. Then

$$\int fg\,dm = \lim_{j\to\infty} \int f_j g_j\,dm \le \lim_{j\to\infty} \int f_j^* g_j^*\,dm \le \int f^* g^*\,dm . \quad ⩟$$

9.3. Remark. This section is from Hardy-Littlewood-Pólya [1, pp. 276-279].

10. Modulus estimates

10.1. We first generalize the concept of the modulus to families of curves which lie on submanifolds of R^n. Let S be an $(n-1)$-dimensional smooth manifold in R^n. In these notes, we need only the cases where S is a sphere or a hyperplane. If Γ is a curve family on S, we again denote by $F(\Gamma)$ the family of all non-negative Borel functions $\rho : S \to \dot{R}^1$ such that $\int_\gamma \rho\,ds \ge 1$ for every locally rectifiable $\gamma \in \Gamma$. The <u>p-modulus of</u> Γ <u>with respect to</u> S is defined by

$$M_p^S(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_S \rho^p \, dm_{n-1} .$$

By a <u>cap</u> of a sphere $S = S^{n-1}(x_0 , r)$ we mean a set $H \cap S$ where H is an open half space in R^n.

10.2. <u>THEOREM.</u> Suppose that $n \geq 2$ and that K is a cap of the sphere $S = S^{n-1}(x_0 , r)$. Suppose also that E and F are disjoint non-empty subsets of $\bar{K}$. Let $\Gamma = \Delta(E , F , K)$. Then

(10.3)
$$M_n^S(\Gamma) \geq b_n/r ,$$

where b_n is the positive constant

(10.4)
$$b_n = 2^{-n-1}\omega_{n-2} \left(\int_0^\infty t^{-\frac{n-2}{n-1}} (1 + t^2)^{-\frac{1}{n-1}} dt \right)^{1-n} , \quad b_2 = 1/2\pi .$$

<u>Proof.</u> Assume first that $n \geq 3$. Since Theorem 8.2 obviously holds also for moduli with respect to a sphere, the inequality (10.3) is invariant under similarity transformations of R^n. Hence we may assume that $S = S^{n-1}(e_n/2 , 1/2)$ and that $e_n \in E$.

Suppose that $\rho \in F(\Gamma)$. Let $f : \bar{R}^n \to \bar{R}^n$ be the inversion in the sphere $S^{n-1}(e_n , 1)$, that is,

(10.5)
$$f(x) = e_n + (x - e_n)/|x - e_n|^2 .$$

Then f is conformal, $f \circ f$ is the identity, and f maps S stereographically onto $\bar{R}^{n-1}$. The image of $S \setminus K$ is either empty or a closed ball or a closed half space in $\bar{R}^{n-1}$. Choose a point $a \in fF$. We may assume that $a = \alpha e_1$, $\alpha \geq 0$. Since $f(S \setminus K)$ is convex, there is an open hemisphere G of S^{n-2} such that $a + ty \in fK$ for every $y \in G$ and $t > 0$. We define $\gamma_y : [0 , \infty] \to S$ by $\gamma_y(t) = f(a+ty)$. Then $\gamma_y \in \Gamma$ for every $y \in G$. Thus

$$1 \leq \int_{\gamma_y} \varrho \, ds = \int_0^\infty \frac{\varrho(f(a+ty))}{1 + |a+ty|^2} \, dt .$$

Integrating over $y \in G$ yields

$$\omega_{n-2}/2 \leq \int_H \frac{\varrho(f(x))}{|x-a|^{n-2}(1+|x|^2)} \, dm_{n-1}(x) ,$$

where H is the half space in R^{n-1}, consisting of all points $a + ty$ where $y \in G$ and $t > 0$. By Hölder's inequality this implies

(10.6) $$(\omega_{n-2}/2)^n \leq$$

$$\int_H \frac{\varrho(f(x))^n}{(1+|x|^2)^{n-1}} \, dm_{n-1}(x) \; (\int_H |x-a|^{-\frac{n(n-2)}{n-1}} (1+|x|^2)^{-\frac{1}{n-1}} \, dm_{n-1}(x) \;)^{n-1} .$$

Since $|f'(x)| = 1/(1 + |x|^2)$ for $x \in R^{n-1}$, we have

(10.7) $$\int_H \frac{\varrho(f(x))^n}{(1+|x|^2)^{n-1}} \, dm_{n-1}(x) = \int_{fH} \varrho^n \, dm_{n-1} \leq \int_S \varrho^n \, dm_{n-1} .$$

We next estimate the second integral I in (10.6). Replacing H by R^{n-1} and using Fubini's theorem we obtain

$$I \leq \int_{R^{n-2}_{1n}} dm_{n-2}(z) \int_{-\infty}^{\infty} |z + (u-\alpha)e_1|^{-\frac{n(n-2)}{n-1}} (1 + |z+ue_1|^2)^{-\frac{1}{n-1}} du ,$$

where $R^{n-2}_{1n} = \{x \mid x_1 = x_n = 0\}$. For each $z \in R^{n-2}_{1n}$ we estimate the inner integral by means of Theorem 9.2. If $z \neq 0$, the functions

$$f_z(u) = |z + (u-\alpha)e_1|^{-\frac{n(n-2)}{n-1}} , \quad g_z(u) = (1 + |z+ue_1|^2)^{-\frac{1}{n-1}}$$

are both bounded, and

$$f_z^*(u) = |z + ue_1|^{-\frac{n(n-2)}{n-1}} , \quad g_z^*(u) = g_z(u) .$$

Hence we obtain

$$I \leq \int_{R^{n-2}_{1n}} dm_{n-2}(z) \int_{-\infty}^{\infty} f_z^* g_z^* \, dm_1 = \int_{R^{n-1}} |z|^{-\frac{n(n-2)}{n-1}} (1 + |z|^2)^{-\frac{1}{n-1}} dm_{n-1}(z) =$$

$$= \omega_{n-2} \int_0^\infty t^{-\frac{n-2}{n-1}} (1+t^2)^{-\frac{1}{n-1}} dt .$$

Together with (10.6) and (10.7) this yields

$$\int_S \varrho^n \, dm_{n-1} \geq 2\, b_n .$$

Since this holds for every $\varrho \in F(\Gamma)$, $M_n^S(\Gamma) \geq 2b_n = b_n/r$.

Next assume that $n = 2$. Then K is an arc of $S^1(x_0, r)$. There is a subarc γ of K such that $\gamma \in \Gamma$. If $\varrho \in F(\Gamma)$, we obtain by Schwarz's inequality

$$1 \leq \left(\int_\gamma \varrho \, ds \right)^2 \leq \int_\gamma \varrho^2 \, ds \int_\gamma ds \leq 2\pi r \int_S \varrho^2 \, dm_1 .$$

Hence $M_2^S(\Gamma) \geq 1/2\pi r = b_2/r$. ΔΔ

10.8. <u>Remark.</u> The integral in (10.4) can be written in several ways. For example,

$$\int_0^\infty t^{-\frac{n-2}{n-1}} (1+t^2)^{-\frac{1}{n-1}} dt = 2(n-1) \int_0^1 (1+t^{2n-2})^{-\frac{1}{n-1}} dt$$

$$= 2^{\frac{n-2}{n-1}} \int_0^{\pi/2} (\sin t)^{-\frac{n-2}{n-1}} dt .$$

The estimate (10.3) is not best possible. However, in the case where the cap K is the whole sphere, we can establish the following sharp result:

10.9. <u>THEOREM.</u> Suppose that $n \geq 2$ and that E and F are disjoint non-empty subsets of the sphere $S = S^{n-1}(x_0, r)$. If $\Gamma = \Delta(E, F, S)$,

(10.10) $$M_n^S(\Gamma) \geq c_n/r ,$$

where

(10.11) $$c_n = 2^n b_n$$

and b_n is given in (10.4). There is equality in (10.10) if $E = \{a\}$, $F = \{b\}$, where a and b are opposite points of S.

Proof. The proof can be carried out as in 10.2. However, the hemisphere G can now be replaced by the whole S^{n-2}. This gives the factor 2^n.

To prove the sharpness, let $S = S^{n-1}(e_n/2, 1/2)$, $E = \{0\}$, $F = \{e_n\}$. Define $\rho : S \to R^1$ by

$$\rho(f(x)) = p_n^{-1} |x|^{-\frac{n-2}{n-1}} (1 + |x|^2)^{\frac{n-2}{n-1}}$$

where p_n is the integral of 10.8 and f is the stereographic projection (10.5). Using 5.7 we see that $\rho \in F(\Gamma)$. Hence

$$M_n^S(\Gamma) \le \int_S \rho^n \, dm_{n-1} = \int_{R^{n-1}} \frac{(\rho(f(x)))^n}{(1 + |x|^2)^{n-1}} \, dm_{n-1}(x) = \omega_{n-2} p_n^{1-n} = c_n/r .$$

The case $n = 2$ needs a separate argument which is left to the reader. ∆∆

10.12. THEOREM. Suppose that $0 < a < b$ and that E and F are disjoint sets such that every sphere $S^{n-1}(t)$, $a < t < b$, meets both E and F. If G contains the spherical ring $A = B^n(b) \setminus \bar{B}^n(a)$ and if $\Gamma = \Delta(E, F, G)$, then

(10.13) $$M(\Gamma) \ge c_n \log \frac{b}{a} ,$$

where c_n is given in (10.11). There is equality in (10.13) if $G = A$ and if E and F are the components of $L \cap A$ where L is a line through the origin.

Proof. Suppose that $\rho \in F(\Gamma)$. For each $a < t < b$ let $\Gamma(t) = \Delta(E \cap S(t), F \cap S(t), S(t))$. Then $\rho | S(t) \in F(\Gamma(t))$. By 10.9 we obtain

$$\int \varrho^n \, dm \geq \int_a^b dt \int_{S(t)} \varrho^n \, dm_{n-1} \geq \int_a^b M_n^{S(t)}(\Gamma(t))\, dt \geq c_n \log \frac{b}{a} .$$

This proves (10.13).

Next assume that $G = A$ and that E and F are the components of $L \cap A$. We may assume that L is the x_n-axis. Let $\Gamma_o = \Delta(e_n, -e_n, S^{n-1})$. From the proof of Theorem 10.9 it follows that there is $\varrho_o \in F(\Gamma_o)$ such that

$$c_n = M_n(\Gamma_o) = \int_{S^{n-1}} \varrho_o^n \, dm_{n-1} .$$

Define $\varrho : R^n \to R^1$ by $\varrho(x) = \varrho_o(x/|x|)/|x|$ for $x \in A$ and $\varrho(x) = 0$ otherwise. We show that $\varrho \in F(\Gamma)$. Let γ be a rectifiable path in Γ and let $f(x) = x/|x|$. Then $f \circ \gamma \in \Gamma_o$, $|f'(x)| = 1/|x|$, and we obtain by 5.4

$$\int_\gamma \varrho \, ds = \int_\gamma \varrho_o(f(x)) \, |f'(x)| \, |dx| \geq \int_{f\circ\gamma} \varrho_o \, ds \geq 1 .$$

Thus $\varrho \in F(\Gamma)$, which implies

$$M(\Gamma) \leq \int \varrho^n \, dm = \int_a^b dt \int_{S(t)} \varrho^n \, dm_{n-1} = \int_a^b \frac{dt}{t} \int_{S^{n-1}} \varrho_o^n \, dm_{n-1} = c_n \log \frac{b}{a} . \quad \Delta\!\!\Delta$$

10.14. Remarks. This section is based on Gehring [3, pp. 355-357]. Theorems 10.2, 10.9 and 10.12 are also true if $E \cap F \neq \emptyset$. In fact, $M_n(\Gamma) = \infty$ in these cases.

11. Rings

11.1. Definition. A domain $A \subset \bar{R}^n$ is a ring if $\underline{C}A$ has exactly two components.

If the components of $\underline{C}A$ are C_o and C_1, we denote

$A = R(C_0, C_1)$. By Topology, ∂A has also two components, namely $B_0 = C_0 \cap \bar{A}$ and $B_1 = C_1 \cap \bar{A}$. To each ring $A = R(C_0, C_1)$ we associate the path family

$$\Gamma_A = \Delta(B_0, B_1, A) .$$

11.2. Remark. We could also consider "rings" which are not necessarily connected, that is, pairs $P = (C_0, C_1)$ of disjoint connected compact sets. To each such pair P we can associate the path family $\Gamma_P = \Delta(C_0, C_1, \bar{R}^n)$. However, from the modulus point of view this is not a generalization. In fact, by Topology there is a unique component A of $\underline{C}(C_0 \cup C_1)$ such that A is a ring with boundary components B_0, B_1 such that $B_i \subset C_i$. Then $\Gamma_A \subset \Gamma_P > \Gamma_A$, whence $M(\Gamma_P) = M(\Gamma_A)$.

On the other hand, we obtain an essential generalization if we allow C_0 and C_1 be non-connected. This generalization is called a condenser, but we will not make use of it in these notes.

11.3. THEOREM. If $A = R(C_0, C_1)$ is a ring with boundary components B_0, B_1 , then the following path families have the same modulus:

$$\begin{aligned}
&\Gamma_A \\
&\Gamma_A^1 = \Delta(C_0, C_1, A) \\
&\Gamma_A^2 = \Delta(C_0, C_1, \bar{R}^n) \\
&\Gamma_A^3 = \Delta(C_0, C_1, R^n) \\
&\Gamma_A^4 = \Delta(B_0, B_1, \bar{R}^n) \\
&\Gamma_A^5 = \Delta(B_0, B_1, R^n) \\
&\Gamma_A^6 = \Delta_0(B_0, B_1, A) .
\end{aligned}$$

Proof. Clearly $\Gamma_A = \Gamma_A^1$. Furthermore, $\Gamma_A^1 \subset \Gamma_A^2 > \Gamma_A^1$ and $\Gamma_A \subset \Gamma_A^4 > \Gamma_A$ imply $M(\Gamma_A^1) = M(\Gamma_A^2)$ and $M(\Gamma_A) = M(\Gamma_A^4)$. Since the paths through ∞ have no influence on the modulus, $M(\Gamma_A^2) = M(\Gamma_A^3)$ and

$M(\Gamma_A^4) = M(\Gamma_A^5)$. Finally, 7.10 implies $M(\Gamma_A^6) = M(\Gamma_A)$. ∆∆

11.4. <u>THEOREM.</u> If $A = R(C_0, C_1)$ and $A' = R(C_0', C_1')$ are rings such that $C_i \subset C_i'$, then $M(\Gamma_A) \leq M(\Gamma_{A'})$.

<u>Proof.</u> Since $\Gamma_A^2 \subset \Gamma_{A'}^2$, the assertion follows from 11.3. ∆∆

11.5. <u>THEOREM.</u> If A is a ring, then $M(\Gamma_A)$ is finite.

<u>Proof.</u> Let $A = R(C_0, C_1)$. We may assume that C_0 is bounded. Choose a positive number $h < d(C_0, C_1)$, and set $E = C_0 + h\bar{B}^n$. Define $\varrho : R^n \to R^1$ by $\varrho(x) = 1/h$ for $x \in E$ and $\varrho(x) = 0$ otherwise. Clearly $\varrho \in F(\Gamma_A)$. Hence

$$M(\Gamma_A) \leq \int \varrho^n \, dm = m(E)/h^n < \infty . \quad ∆∆$$

If A is the spherical ring $a < |x| < b$, $M(\Gamma_A) = \omega_{n-1}(\log \frac{b}{a})^{1-n}$ by 7.5. In the rest of this section we derive estimates for the moduli of more general rings.

11.6. <u>Definition.</u> Given $r > 0$, we let $\Phi_n(r)$ be the set of all rings $A = R(C_0, C_1)$ in $\bar{R}^n$ with the following properties: (1) C_0 contains the origin and a point a such that $|a| = 1$. (2) C_1 contains ∞ and a point b such that $|b| = r$. We denote

$$\mathcal{H}_n(r) = \inf M(\Gamma_A)$$

over all rings $A \in \Phi_n(r)$.

By 11.5, $\mathcal{H}_n(r)$ is a non-negative finite number.

11.7. <u>THEOREM.</u> The function $\mathcal{H}_n : (0, \infty) \to R^1$ has the following properties:

(1) $\mathscr{H}_n$ is decreasing.

(2) $\lim_{r\to\infty} \mathscr{H}_n(r) = 0$.

(3) $\lim_{r\to 0} \mathscr{H}_n(r) = \infty$.

(4) $\mathscr{H}_n(r) > 0$ for every $r > 0$.

<u>Proof.</u> (1): If $r < s$, then $\Phi_n(r) \subset \Phi_n(s)$, which implies $\mathscr{H}_n(r) \geq \mathscr{H}_n(s)$.

(2): For $r > 0$ let $A(r)$ be the spherical ring $\{x \mid 1 < |x + e_1| < 1 + r\}$. Then $A(r) \in \Phi_n(r)$, which implies

$$\mathscr{H}_n(r) \leq M(\Gamma_{A(r)}) = \omega_{n-1} (\log(1+r))^{1-n} \to 0$$

as $r \to \infty$.

(3): Assume that $r < 1$ and that $A \in \Phi_n(r)$. Since the sphere $S(t)$ meets the boundary components of A for $r < t < 1$, 10.12 and 11.3 imply $M(\Gamma_A) = M(\Gamma_A^2) \geq c_n \log \frac{1}{r}$, which proves (3).

(4): Since $\mathscr{H}_n$ is decreasing, we may assume that $r \geq 1$. (The case $r < 1$ was in fact considered in the proof of (3).) Let $A = R(C_0, C_1) \in \Phi_n(r)$, and let $a \in C_0$ and $b \in C_1$ be such that $|a| = 1$, $|b| = r$. Let α be the angle $(0, a/2, b)$, $0 \leq \alpha \leq \pi$. We divide the rest of the proof in two cases according as α is acute or not.

<u>Case 1.</u> $0 \leq \alpha \leq \pi/2$. Since C_0 and C_1 are connected, $S(b/2, t)$ meets both C_0 and C_1 for $r/2 < t < |a - b/2|$. By 10.12 and 11.3 this implies

$$M(\Gamma_A) \geq c_n \log \frac{|a - b/2|}{r/2}.$$

Using elementary geometry it is easy to see that $|a - b/2|$ attains its smallest value $(r^2 + 2)^{1/2}/2$ for $\alpha = \pi/2$. Consequently,

$$M(\Gamma_A) \geq \frac{c_n}{2} \log (1 + 2/r^2). \tag{11.8}$$

<u>Case 2.</u> $\pi/2 < \alpha \leq \pi$. Now the spheres $S((a+b)/2, t)$ meet C_0 and C_1 for $|a-b|/2 < t < |a+b|/2$. Again by 10.12 we obtain

$$M(\Gamma_A) \geq c_n \log \frac{|a+b|}{|a-b|}.$$

Here $|a+b|/|a-b|$ attains its smallest value $(1+2/r^2)^{1/2}$ for $\alpha = \pi/2$, and we again obtain the inequality (11.8). Hence

$$\mathscr{H}_n(r) \geq \frac{c_n}{2} \log (1+2/r^2) > 0$$

for $r \geq 1$. ∆∆

From the definition of $\mathscr{H}_n$ and from the conformal invariance of the modulus we obtain the following estimate:

11.9. <u>THEOREM.</u> Suppose that $A = R(C_0, C_1)$ is a ring and that $a, b \in C_0$ and $c, \infty \in C_1$. Then

$$M(\Gamma_A) \geq \mathscr{H}_n\left(\frac{|c-a|}{|b-a|}\right). \ ∆∆$$

11.10. <u>THEOREM.</u> Suppose that $A = R(C_0, C_1)$ is a ring. Then $M(\Gamma_A) = 0$ iff C_0 or C_1 consists of a single point.

<u>Proof.</u> If C_0 or C_1 is a point, then $M(\Gamma_A) = 0$ by 7.9. Suppose next that both C_0 and C_1 are non-degenerate. Then we can choose distinct points $a, b \in C_0$ and $c, d \in C_1$. By performing an auxiliary Möbius transformation, we may assume that $d = \infty$. Then $M(\Gamma_A) > 0$ by 11.9 and 11.7.(4). ∆∆

11.11. <u>Remarks.</u> 1. There is an alternate way to define $M(\Gamma_A)$, due to Loewner [1] and Gehring [2]. Let $A = R(C_0, C_1)$, and consider all C^1-functions $u : A \to R^1$ such that $u(x) \to 0$ as $x \to C_0$ and $u(x) \to 1$ as $x \to C_1$. Then

$$M(\Gamma_A) = \inf \int_A |\operatorname{grad} u|^n \, dm$$

over all such functions u. A third way to define $M(\Gamma_A)$ is to consider the family of all (n-1)-dimensional surfaces in A, separating C_0 from C_1. The equivalence of these three definitions has been proved by Gehring [4]. See also Ziemer [1].

2. The proof of 11.7 is from Väisälä [1]. Stronger results have been proved by Gehring [2] who showed that $\mathcal{H}_n(r) = M(\Gamma_{A(r)})$ where $A(r)$ is the so-called Teichmüller ring $R(C_0, C_1)$, $C_0 = \{te_1 \mid -1 \leq t \leq 0\}$ and $C_1 = \{te_1 \mid r \leq t \leq \infty\}$. The proof of this important and deep result is based on the spherical symmetrization of rings. From it one can also conclude that $\mathcal{H}_n$ is strictly decreasing.

12. Modulus estimates in the spherical metric

The euclidean distance does not define a metric in $\bar{R}^n$, since the distance between ∞ and a finite point is not defined. In this section we define the so-called spherical metric in $\bar{R}^n$ and derive some modulus estimates. These will not be needed until in Section 19.

12.1. Definition. The spherical (chordal) distance between two points $a, b \in \bar{R}^n$ is the number

$$q(a, b) = |f(a) - f(b)|,$$

where $f : \bar{R}^n \to S^n(e_{n+1}/2, 1/2)$ is the stereographic projection, defined by

$$f(x) = e_{n+1} + \frac{x - e_{n+1}}{|x - e_{n+1}|^2}.$$

Since f is a homeomorphism, q is a metric in $\bar{R}^n$ and defines its usual topology. Explicitly, if $a \neq \infty \neq b$,

$$q(a, b) = |a - b|(1 + |a|^2)^{-1/2}(1 + |b|^2)^{-1/2},$$

and

$$q(a, \infty) = (1 + |a|^2)^{-1/2}.$$

Furthermore, we have always $q(a, b) \leq |a - b|$ and $q(a, b) \leq 1$. In the usual way, we define the spherical distance $q(A, B)$ between two sets A, B and the spherical diameter $q(A)$ of a set A.

12.2. THEOREM. Given two points $a, b \in \bar{R}^n$, there is a Möbius transformation $g : \bar{R}^n \to \bar{R}^n$ such that $g(a) = b$ and such that g preserves all spherical distances.

Proof. The mapping g has the form $g = f \circ h \circ f$ where f is the stereographic projection $(f = f^{-1})$ and h is a suitable rotation of the sphere $S^n(e_{n+1}, 1/2)$. ΔΔ

12.3. Definition. The mapping g of 12.2 is called a spherical isometry.

12.4. Definition. Given $0 < r \leq 1$, we let $\Psi_n(r)$ be the set of all rings $A = R(C_0, C_1)$ in $\bar{R}^n$ with the following properties: (1) $q(C_0) \geq r$, (2) $q(C_1) \geq r$. We denote

$$\lambda_n(r) = \inf M(\Gamma_A)$$

over all rings $A \in \Psi_n(r)$.

12.5. THEOREM. The function $\lambda_n : (0, 1] \to R^1$ has the following properties:

(1) λ_n is increasing.

(2) $\lim_{r \to 0} \lambda_n(r) = 0$.

(3) $\lambda_n(r) > 0$ for every $0 < r \leq 1$.

Proof. (1): If $r < s$, then $\Psi_n(r) \supset \Psi_n(s)$, which implies $\lambda_n(r) \leq \lambda_n(s)$.

(2): This simply states that there are rings A, e.g. spherical rings, such that the boundary components of A are non-degenerate and such that $M(\Gamma_A)$ is arbitrarily small.

(3): Suppose that $A = R(C_0, C_1) \in \Psi_n(r)$. Choose $a, b \in C_0$ and $c, d \in C_1$ such that $q(a, b) \geq r$ and $q(c, d) \geq r$. Performing a spherical isometry of $\bar{R}^n$ we may assume that $d = \infty$. By 11.9, $M(\Gamma_A) \geq \mathcal{H}_n(|c-a|/|b-a|)$. We next estimate $|c-a|/|b-a|$. Assuming that $|a| \leq |b|$ we obtain

$$r \leq q(a, b) = |a-b|(1+|a|^2)^{-1/2}(1+|b|^2)^{-1/2} \leq |a-b|/(1+|a|^2)$$

and

$$r \leq q(c, \infty) = (1+|c|^2)^{-1/2} \leq 1/|c| .$$

Hence

$$\frac{|c-a|}{|b-a|} \leq \frac{1+r|a|}{r^2(1+|a|^2)} \leq u(r) ,$$

where

$$u(r) = \max_{0 \leq t < \infty} \frac{1+rt}{r^2(1+t^2)} < \infty .$$

Since $\mathcal{H}_n$ is increasing, $M(\Gamma_A) \geq \mathcal{H}_n(u(r))$. Hence $\lambda_n(r) \geq \mathcal{H}_n(u(r)) > 0$. ΔΔ

We next consider a more general problem.

12.6. Definition. Given $0 < r \leq 1$ and $0 < t \leq 1$, we let $\Psi_n(r, t)$ be the set of all rings $A = R(C_0, C_1)$ in $\bar{R}^n$ with the following properties: (1) $q(C_0) \geq r$, (2) $q(C_1) \geq r$, (3) $q(C_0, C_1) \leq t$. We denote

$$\lambda_n(r, t) = \inf M(\Gamma_A)$$

over all rings $A \in \Psi_n(r, t)$.

Thus $\lambda_n(r, 1)$ is equal to the number $\lambda_n(r)$, defined in 12.4.

12.7. THEOREM. The function $\lambda_n : (0, 1] \times (0, 1] \to R^1$ has the following properties:

(1) $\lambda_n(r, t)$ is increasing in r.

(2) $\lambda_n(r, t)$ is decreasing in t.

(3) $\lambda_n(r, t) \geq \lambda_n(r) > 0$ for every r and t.

(4) $\lim_{t \to 0} \lambda_n(r, t) = \infty$ for every r.

Proof. (1) and (2) are obvious. (3) follows from (2) and 12.5. To prove (4), suppose that $t < r/4$ and that $A = R(C_0, C_1) \in \Psi_n(r, t)$. Pick $a \in C_0$ and $c \in C_1$ such that $q(a, c) \leq t$. Next choose $b \in C_0$ and $d \in C_1$ such that $q(a, b) \geq r/2$ and $q(c, d) \geq r/2$. Performing an auxiliary spherical isometry, we may assume that $d = \infty$. Hence $M(\Gamma_A) \geq \mathcal{H}_n(|c-a|/|b-a|)$. Since $1/|c| \geq q(c, \infty) \geq r/2$ and since $1/|a| \geq q(a, \infty) \geq q(c, \infty) - q(a, c) \geq r/2 - t \geq r/4$, we obtain

$$|c - a| = q(a, c)(1 + |a|^2)^{1/2}(1 + |c|^2)^{1/2} \leq t(1 + 16/r^2) .$$

On the other hand, $|b - a| \geq q(a, b) \geq r/2$, whence

$$M(\Gamma_A) \geq \mathcal{H}_n(\frac{2t}{r}(1 + 16/r^2)) .$$

Hence $\lambda_n(r, t) \geq \mathcal{H}_n(\frac{2t}{r}(1+16/r^2))$ for $t < r/4$. Since $\mathcal{H}_n(s) \to \infty$ as $s \to 0$, this proves (4). Δ

CHAPTER 2. QUASICONFORMAL MAPPINGS

In this chapter we give the definition of qc mappings and establish various results concerning the boundary behavior, distortion and convergence of qc mappings. The only tools we need are the results of Chapter 1 and some topological properties of R^n. The chapter consists of sections 13-22.

13. The dilatations of a homeomorphism

Suppose that $f : D \to D'$ is a homeomorphism. As mentioned in "Notations", this includes the assumption that D and D' are domains in $\bar{R}^n$. Consider a path family Γ in D and its image family $\Gamma' = \{ f \circ \gamma \mid \gamma \in \Gamma \}$. If f is conformal, $M(\Gamma') = M(\Gamma)$ by 8.1. Hence it seems natural to introduce the quantities

$$K_I(f) = \sup \frac{M(\Gamma')}{M(\Gamma)} , \qquad K_O(f) = \sup \frac{M(\Gamma)}{M(\Gamma')} ,$$

where the suprema are taken over all path families Γ in D such that $M(\Gamma)$ and $M(\Gamma')$ are not simultaneously 0 or ∞. It is clear that there are such families, e.g. those associated to a ring with non-degenerate boundary components. If f is conformal, $K_I(f) = K_O(f) = 1$. We can regard these numbers as measures for how much f differs from a conformal mapping.

Since $M(\Gamma) = M(\Gamma') = \infty$ whenever one (and hence the other) of the families Γ and Γ' contains a constant path, the constant paths have no influence on $K_I(f)$ and $K_O(f)$. In order to avoid technical difficulties, we shall therefore assume from now on that <u>every path family contains only non-constant paths</u>.

13.1. <u>Definition.</u> If $f : D \to D'$ is a homeomorphism, $K_I(f)$ is the <u>inner dilatation</u> and $K_O(f)$ is the <u>outer dilatation</u> of f. The <u>maximal dilatation</u> of f is $K(f) = \max(K_I(f), K_O(f))$. If $K(f) \leq K < \infty$, f is <u>K-quasiconformal</u>. Equivalently, f is K-quasiconformal iff

$$M(\Gamma)/K \leq M(\Gamma') \leq K\, M(\Gamma)$$

for every path family Γ in D. f is <u>quasiconformal</u> (abbreviated qc) if $K(f) < \infty$.

It follows from the definitions that the dilatations are positive numbers, possibly infinite. We shall show in 34.5 that they are always ≥ 1. For the moment we can only say that either $K_I(f) \geq 1$ or $K_O(f) \geq 1$. Hence $K(f) \geq 1$.

From the definition of the dilatations we readily obtain the following relations:

13.2. <u>THEOREM.</u> (1) $K_I(f^{-1}) = K_O(f)$.

(2) $K_O(f^{-1}) = K_I(f)$.

(3) $K(f^{-1}) = K(f)$.

(4) $K_I(f \circ g) \leq K_I(f)\, K_I(g)$.

(5) $K_O(f \circ g) \leq K_O(f)\, K_O(g)$.

(6) $K(f \circ g) \leq K(f)\, K(g)$. ΔΔ

13.3. <u>COROLLARY.</u> If f is K-qc, then f^{-1} is K-qc. ΔΔ

13.4. <u>COROLLARY.</u> If $h = f \circ g$, where f is K_1-qc and g is K_2-qc, then h is K_1K_2-qc. ΔΔ

13.5. <u>COROLLARY.</u> If $h = g_1 \circ f \circ g_2$, where g_1 and g_2 are conformal, then h and f have the same dilatations. ⩕

13.6. <u>Definition.</u> If D can be mapped qclly onto D', D is <u>quasiconformally equivalent</u> to D'.

By 13.3 and 13.4, the above relation is really an equivalence relation.

13.7. <u>Remarks.</u> 1. There are several equivalent ways to define the dilatations of a homeomorphism. We shall give two other definitions in 34.4 and 36.1.

2. Gehring [3] and Rešetnjak [2] have proved the following strong form of Liouville's theorem 5.8: If $n \geq 3$ and if $f : D \to D'$ is 1-qc, then f is a restriction of a Möbius transformation. We shall not prove this important result in these notes. Mostow [1] has given a rather elementary proof for the fact that a 1-qc mapping of $\bar{R}^n$ onto $\bar{R}^n$ is a Möbius transformation. This result is also true for $n = 2$.

14. The dilatations of a linear mapping

14.1. <u>Definition.</u> Let $A : R^n \to R^n$ be a linear bijection. The numbers

$$H_I(A) = \frac{|\det A|}{\ell(A)^n}, \quad H_O(A) = \frac{|A|^n}{|\det A|}, \quad H(A) = \frac{|A|}{\ell(A)}$$

are called the <u>inner</u>, <u>outer</u>, and <u>linear dilatation</u> of A, respectively. For notation, see p. viii.

In Section 15 we show that $H_I(A) = K_I(A)$ and $H_O(A) = K_O(A)$, which justifies the terminology.

Obviously, all three dilatations are ≥ 1. They have the following geometric interpretation: The image of the unit ball B^n under A is an ellipsoid $E(A)$. Let $B_I(A)$ and $B_O(A)$ be the inscribed and circumscribed balls of $E(A)$, respectively. Then

$$H_I(A) = \frac{m(E(A))}{m(B_I(A))}, \qquad H_O(A) = \frac{m(B_O(A))}{m(E(A))},$$

and $H(A)$ is the ratio of the greatest and the smallest semiaxis of $E(A)$.

Let $a_1 \geq a_2 \geq \dots \geq a_n$ be the semi-axes of $E(A)$. More precisely, the numbers a_i are positive square roots of the proper values of A^*A where A^* is the adjoint of A. Then $a_1 = |A|$, $a_n = \ell(A)$, $|\det A| = a_1 \dots a_n$, and we can also write

$$(14.2)\quad H_I(A) = \frac{a_1 \dots a_{n-1}}{a_n^{n-1}}, \quad H_O(A) = \frac{a_1^{n-1}}{a_2 \dots a_n}, \quad H(A) = \frac{a_1}{a_n}.$$

If $n = 2$, then $H_I(A) = H_O(A) = H(A)$. In the general case we obtain the following relations:

$$H_I(A) \leq H_O(A)^{n-1}, \quad H_O(A) \leq H_I(A)^{n-1}, \quad H(A)^n = H_I(A)H_O(A),$$

(14.3)

$$H(A) \leq \min(H_I(A), H_O(A)) \leq H(A)^{n/2} \leq \max(H_I(A), H_O(A)) \leq H(A)^{n-1}.$$

It is easy to see that these inequalities are best possible. For example, if $a_1 = \dots = a_{n-1} > a_n$, then $H_I(A) = H_O(A)^{n-1} = H(A)^{n-1}$. The following relations are easy consequences of the definitions:

$$H_I(A^{-1}) = H_O(A), \quad H_O(A^{-1}) = H_I(A), \quad H(A^{-1}) = H(A),$$

(14.4)

$$H_I(AB) \leq H_I(A)H_I(B), \quad H_O(AB) \leq H_O(A)H_O(B), \quad H(AB) \leq H(A)H(B).$$

If A is given by its matrix, it is usually an elaborate task to compute the dilatations of A. We show, however, that in the case $n = 2$ this can be done by a relatively simple formula.

We use complex notations in R^2 and write $z = x + iy$. Let $A : R^2 \to R^2$ be a linear bijection. Then $Az = ax + by + i(cx + dy)$, where a, b, c, d are real numbers such that $ad - bc \neq 0$. Set $H = H_I(A) = H_O(A) = H(A)$. The problem is to compute H in terms of a, b, c, d.

Let $z = e^{i\varphi}$ be a unit vector. Then $2Az = pe^{i\varphi} + qe^{-i\varphi}$ where $p = a + d + i(c - b)$, $q = a - d + i(c + b)$. Consequently, $2|Az| = |p + qe^{-2i\varphi}|$. From this we see that $2|A| = |p| + |q|$ and $2\ell(A) = ||p| - |q||$, whence

$$H = \frac{|p| + |q|}{||p| - |q||} .$$

This formula can be used for the calculation of H. From it we also easily obtain

$$H + \frac{1}{H} = \frac{a^2 + b^2 + c^2 + d^2}{|ad - bc|} . \tag{14.5}$$

Thus H is the greater root of this equation (the other root is $1/H$).

For higher dimensions, we only give a pair of inequalities which can be used in numerical estimates. Suppose that $A : R^n \to R^n$ is a linear bijection and that (a_{ij}) is its matrix, i.e., $Ae_i = \sum_j a_{ji} e_j$. If $|x| \leq 1$, then

$$|Ax|^2 = \sum_j \Big(\sum_i a_{ji} x_i \Big)^2 \leq \sum_j \Big(\sum_i a_{ji}^2 \Big) \Big(\sum_i x_i^2 \Big) \leq \sum_{i,j} a_{ij}^2 .$$

Hence,

$$H_O(A) \leq \frac{\left(\sum a_{ij}^2 \right)^{n/2}}{|\det A|} . \tag{14.6}$$

In the other direction, we have

$$\sum_{i,j} a_{ij}^2 = \sum_i |Ae_i|^2 \leq n |A|^2 ,$$

whence

$$H_O(A) \geq n^{-n/2} \frac{\left(\sum a_{ij}^2 \right)^{n/2}}{|\det A|} . \tag{14.7}$$

15. Quasiconformal diffeomorphisms

Suppose that we are given a homeomorphism $f: D \to D'$. How can we determine the dilatations $K_I(f)$ and $K_O(f)$? In particular, how can we find out whether f is qc or not? This problem is usually impossible to solve directly using the definition of the dilatations, since we ought to compute the moduli of all path families in D and D'. In this section we show how the dilatations of a diffeomorphism can be calculated in terms of its derivative. For more general mappings, this problem is considered in Section 34.

By a diffeomorphism we mean a homeomorphism $f: D \to D'$, where D and D' are domains in R^n, such that both f and f^{-1} belong to C^1. Equivalently, a diffeomorphism is a C^1-homeomorphism whose jacobian $J(x, f)$ does not vanish. If f is a diffeomorphism,

$$H_O(f'(x)) = \frac{|f'(x)|^n}{|J(x, f)|}, \qquad H_I(f'(x)) = \frac{|J(x, f)|}{\ell(f'(x))^n}.$$

15.1. THEOREM. Suppose that $f: D \to D'$ is a diffeomorphism. Then

$$K_I(f) = \sup_{x \in D} H_I(f'(x)), \qquad K_O(f) = \sup_{x \in D} H_O(f'(x)).$$

Proof. It suffices to prove the second equation, because the first one follows from it if we consider the inverse mapping f^{-1}. We first show that $K_O(f) \leq \sup H_O(f'(x))$. We may assume that $\sup H_O(f'(x)) = K < \infty$. We have to prove that $M(\Gamma) \leq K M(\Gamma')$ for an arbitrary path family Γ in D. The proof is closely similar to the proof for the conformal invariance 8.1 of the modulus. In fact, 8.1 is a special case of 15.1.

Let $\varrho' \in F(\Gamma')$. Define $\varrho: R^n \to \dot{R}^1$ by $\varrho(x) = \varrho'(f(x))\,|f'(x)|$

for $x \in D$ and $\varrho(x) = 0$ otherwise. We show that $\varrho \in F(\Gamma)$. Suppose that γ is a locally rectifiable path in Γ. Using 5.4 we obtain

$$\int_\gamma \varrho \, ds \geq \int_{f\circ\gamma} \varrho' \, ds \geq 1 .$$

Hence $\varrho \in F(\Gamma)$, which implies

$$M(\Gamma) \leq \int \varrho^n \, dm = \int_D \varrho'(f(x))^n \, |f'(x)|^n \, dm(x)$$

$$\leq K \int_D \varrho'(f(x))^n \, |J(x, f)| \, dm(x) = K \int_{D'} \varrho'^n \, dm \leq K \int \varrho'^n \, dm .$$

Since this holds for every $\varrho' \in F(\Gamma')$, we obtain $M(\Gamma) \leq K \, M(\Gamma')$.

It remains to prove that $H_0(f'(x)) \leq K_0(f)$ for every $x \in D$. We prove a more general result which will be needed in Section 32.

15.2. <u>THEOREM.</u> Let $f : D \to D'$ be a homeomorphism. If f is differentiable at a point $a \in D$ and if $K_0(f) < \infty$, then

$$|f'(a)|^n \leq K_0(f) \, |J(a, f)| .$$

<u>Proof.</u> By performing a preliminary similarity transformation, we may assume that $a = 0 = f(a)$ and that $f'(0)$ is given by $f'(0)e_i = a_i e_i$ where $a_1 \geq \ldots \geq a_n \geq 0$. We have to show that

$$a_1^n \leq K_0(f) \, a_1 \ldots a_n . \tag{15.3}$$

Since (15.3) is trivial if $a_1 = 0$, we may assume $a_1 > 0$. Let $0 < \varepsilon < a_1/2$ and let $Q = \{x \in R^n \mid 0 < x_i < \delta\}$ be a cube such that $\bar{Q} \subset D$ and such that $|f(x) - f'(0)x| \leq \varepsilon\delta$ for $x \in \bar{Q}$. Let E and F be the faces $x_1 = 0$ and $x_1 = \delta$ of Q, respectively. Set $\Gamma = \Delta(E, F, Q)$. By 7.2, $M(\Gamma) = 1$. We next estimate $M(\Gamma')$. Since fE lies between the hyperplanes $x_1 = \pm\varepsilon\delta$ and since fF lies between the hyperplanes $x_1 = (a_1 \pm \varepsilon)\delta$, $\ell(\gamma) \geq (a_1 - 2\varepsilon)\delta$ for every $\gamma \in \Gamma'$. On the other hand, fQ is contained in the n-interval $\{x \mid -\varepsilon\delta \leq x_i \leq (a_i + \varepsilon)\delta\}$. Hence 7.1 implies $M(\Gamma') \leq (a_1 + 2\varepsilon) \ldots (a_n + 2\varepsilon)(a_1 - 2\varepsilon)^{-n}$, whence

$$(a_1 - 2\varepsilon)^n \leq \frac{M(\Gamma)}{M(\Gamma')} (a_1 + 2\varepsilon)\ldots(a_n + 2\varepsilon) .$$

Since $M(\Gamma) \leq K_O(f)\, M(\Gamma')$ and since $\varepsilon > 0$ is arbitrary, this implies (15.3). ∆∆

We give some consequences of Theorem 15.1.

15.4. <u>COROLLARY.</u> A diffeomorphism $f : D \to D'$ is K-qc iff the double inequality

$$|f'(x)|^n/K \leq |J(x, f)| \leq K\, \ell(f'(x))^n$$

holds for every $x \in D$. ∆∆

15.5. <u>COROLLARY.</u> If $f : D \to D'$ is a diffeomorphism, then

$$1 \leq K_I(f) \leq K_O(f)^{n-1}, \qquad 1 \leq K_O(f) \leq K_I(f)^{n-1}.$$

These inequalities follow from (14.3). ∆∆ They are true for all homeomorphisms although not proved until in Section 34. From 15.2 we also obtain:

15.6. <u>COROLLARY.</u> If a qc mapping f is differentiable at a point a, then either $f'(a) = 0$ or $J(a, f) \neq 0$. ∆∆

16. Examples

In this section we give some examples of qc mappings.

16.1. <u>Linear mapping.</u> Let $A : R^n \to R^n$ be a linear bijection. Then $A'(x) = A$ for all $x \in R^n$. From 15.1 we obtain

$$K_I(A) = H_I(A), \qquad K_O(A) = H_O(A).$$

Thus A is qc.

16.2. A radial mapping. Let $a \neq 0$ be a real number, and set $f(x) = |x|^{a-1} x$. Then f is a diffeomorphism of $R^n \setminus \{0\}$ onto itself. We can extend f to a homeomorphism $f^* : \bar{R}^n \to \bar{R}^n$ by defining $f^*(0) = 0$, $f^*(\infty) = \infty$ if $a > 0$, and $f^*(0) = \infty$, $f^*(\infty) = 0$ if $a < 0$. We compute the dilatations of f and f^*.

It is easy to see, by symmetry, that the dilatation ellipsoid $E(f'(x))$ has semi-axes $|a||x|^{a-1}$, $|x|^{a-1}$, ..., $|x|^{a-1}$. Hence (14.2) and 15.1 imply

$$K_I(f) = |a|, \qquad K_O(f) = |a|^{n-1} \quad \text{if } |a| \geq 1.$$

$$K_I(f) = |a|^{1-n}, \qquad K_O(f) = |a|^{-1} \quad \text{if } |a| \leq 1.$$

Thus f is qc.

We next show that f^* has the same dilatations as f. Let Γ be a path family in $\bar{R}^n$. Set $\Gamma_0 = \{\gamma \in \Gamma \mid 0 \in |\gamma|\}$, $\Gamma_\infty = \{\gamma \in \Gamma \mid \infty \in |\gamma|\}$, and $\Gamma_1 = \Gamma \setminus (\Gamma_0 \cup \Gamma_\infty)$. By 7.9, $M(\Gamma_0) = M(\Gamma_\infty) = M(\Gamma_0') = M(\Gamma_\infty') = 0$. Hence $M(\Gamma) = M(\Gamma_1)$ and $M(\Gamma') = M(\Gamma_1')$. Thus $K_I(f^*) = K_I(f)$ and $K_O(f^*) = K_O(f)$.

If $a = -1$, f is conformal. It is the inversion in the unit sphere S^{n-1}.

16.3. Folding. Let (r, φ, z) be the cylindrical coordinates of a point $x \in R^n$. This means that $r \geq 0$, $0 \leq \varphi < 2\pi$, $z \in R^{n-2}$, and

$$x_1 = r\cos\varphi,$$
$$x_2 = r\sin\varphi,$$
$$x_i = z_{i-2} \quad \text{for } 3 \leq i \leq n.$$

The domain D_α, defined by $0 < \varphi < \alpha$, is called a wedge of angle α, $0 < \alpha \leq 2\pi$. Given two wedges D_α and D_β, we define a homeomorphism $f : D_\alpha \to D_\beta$ by $f(r, \varphi, z) = (r, \beta\varphi/\alpha, z)$. This mapping is called a <u>folding</u>.

A folding is clearly a diffeomorphism. Moreover, the semi-axes of $E(f'(x))$ are $\beta/\alpha, 1, \dots, 1$. Assuming that $\alpha \leq \beta$ we thus have by 15.1,

$$K_I(f) = \beta/\alpha, \qquad K_O(f) = (\beta/\alpha)^{n-1}.$$

Thus f is always qc. In particular, choosing $\alpha = \pi, \beta = 2\pi$, we see that a half space can be mapped qclly onto a domain whose exterior is empty.

16.4. <u>Cones.</u> Let (R, φ, θ) be the spherical coordinates of a point $x \in R^3$. This means that $R \geq 0$, $0 \leq \varphi < 2\pi$, $0 \leq \theta \leq \pi$, and

$$x_1 = R \sin\theta \cos\varphi,$$
$$x_2 = R \sin\theta \sin\varphi,$$
$$x_3 = R \cos\theta.$$

The domain C_α, defined by $\theta < \alpha$, is called a <u>cone</u> of angle α, $0 < \alpha \leq \pi$. A natural homeomorphism $f : C_\alpha \to C_\beta$ is defined by $f(R, \varphi, \theta) = (R, \varphi, \beta\theta/\alpha)$. Assuming $\alpha \leq \beta$ we have $K_O(f) = \beta^2 \sin\alpha / \alpha^2 \sin\beta$. The inner dilatation $K_I(f)$ is equal to $\max(\beta^2/\alpha^2, \beta \sin^2\alpha / \alpha \sin^2\beta)$. The proof is left as an exercise to the reader. Thus f is qc if $\beta < \pi$. If $\beta = \pi$, f is not qc. Thus the "natural" homeomorphism of a half space onto the complement of a ray is not qc. We shall show in 17.23 that these domains are not qclly equivalent.

16.5. <u>The infinite cylinder.</u> Let (R, φ, θ) again be the spherical coordinates in R^3, and let D be the half space $\theta < \pi/2$. We define a mapping f of D as follows: The cylindrical coordinates of $f(R, \varphi, \theta)$ are $r = \theta$, $\varphi = \varphi$, $z = \log R$. Then f is a diffeomorphism of D onto the infinite cylinder $r < \pi/2$. This mapping is qc with $K_I(f) = \pi/2$, $K_O(f) = \pi^2/4$. The proof is again left to the reader.

16.6. Projection. Denote by P the orthogonal projection of R^n onto R^{n-1}. Let D_o be a domain in R^{n-1}. Then $D = P^{-1}D_o$ is a domain in R^n. Suppose that $u : D_o \to R^1$ is a C^1-function. Define $f : D \to D$ by $f(x) = x - u(P(x))\, e_n$. Then f is a diffeomorphism which maps the $(n-1)$-manifold $x_n = u(x_1, \ldots, x_{n-1})$ onto D_o. We compute the dilatations of f.

Assume first that $n = 2$. Then we obtain from (14.5)

$$H(f'(x)) = (|u'(Px)| + (|u'(Px)|^2 + 4)^{1/2})^2 / 4 .$$

Since $J(x, f) = 1$, $|f'(x)|^2 = H(f'(x)) = 1/\ell(f'(x))^2$.

Next consider the general case. Again $J(x, f) = 1$. Considering restrictions of f to 2-dimensional planes parallel to e_n we see that

$$|f'(x)| = \ell(f'(x))^{-1} = (|u'(Px)| + (|u'(Px)|^2 + 4)^{1/2}) / 2 .$$

Hence

$$K_I(f) = K_O(f) = 2^{-n}(a + (a^2+4)^{1/2})^n ,$$

where

$$a = \sup_{x \in D_o} |u'(x)| .$$

Thus f is qc iff $a < \infty$.

17. Boundary extension

In this section we study under which conditions a qc mapping $f : D \to D'$ has a limit at a boundary point of D. If this limit exists in a set $A \subset \partial D$, we also give some conditions under which the extension of f to $D \cup A$ is a homeomorphism. For example, we show that a qc mapping $f : B^n \to D'$ can be extended to a homeomorphism of $\bar{B}^n$ onto $\bar{D}'$ iff $\partial D'$ is homeomorphic to ∂B^n.

17.1. We first give some topological notions. Given a mapping $f : D \to \bar{R}^n$ and a point $b \in \partial D$, the <u>cluster set</u> $C(f, b)$ of f at b is the set of all points $b' \in \bar{R}^n$ such that there exists a sequence $x_1, x_2, \ldots$ such that $x_j \in D$, $x_j \to b$ and $f(x_j) \to b'$. Alternatively, $C(f, b) = \bigcap \overline{f(D \cap U)}$ where U runs through all neighborhoods ob b. Thus f has a limit b' at b iff $C(f, b) = \{b'\}$. Since $\bar{R}^n$ is compact, the cluster set is never empty. The cluster set of f on a set $A \subset \partial D$ is defined by

$$C(f, A) = \bigcup_{b \in A} C(f, b) .$$

The cluster sets of a homeomorphism $f : D \to D'$ are always subsets of $\partial D'$.

17.2. We also need the following topological results: If D is a domain in $\bar{R}^n$ and if C_i, $i \in I$, are the components of $\complement D$, then $\bar{D} \cap C_i$, $i \in I$, are the components of ∂D. If $f : D \to D'$ is a homeomorphism and if B is a component of ∂D, then $C(f, B)$ is a component of $\partial D'$. Hence D and D' have the same number of boundary components. In particular, the homeomorphic image of a ring is always a ring.

Our first result states that isolated boundary points are removable singularities.

17.3. <u>THEOREM.</u> Suppose that $f : D \to D'$ is a qc mapping and that b is an isolated point of ∂D. Then f has a limit b' at b, and b' is an isolated point of $\partial D'$. Defining $f^*(b) = b'$ and $f^* | D = f$ we obtain a qc mapping $f^* : D \cup \{b\} \to D' \cup \{b'\}$. Moreover,

$K_I(f^*) = K_I(f)$ and $K_O(f^*) = K_O(f)$.

Proof. Choose a ball neighborhood U of b such that $\bar{U} \cap \partial D = \{b\}$. Then $A = U \setminus \{b\}$ is a ring and $M(\Gamma_A) = 0$. fA is also a ring with boundary components $f\partial U$ and $C(f, b)$. Since f is qc, $M(\Gamma_{fA}) = 0$. (We use the family Γ_A^6 of 11.3.) By 11.10, $C(f, b)$ consists of a single point b'. Since $q(b', \partial D' \setminus \{b'\}) \geq q(b', f\partial U) > 0$, b' is an isolated point of $\partial D'$. The set $D \cup \{b\}$ is clearly a domain, and $f^* : D \cup \{b\} \to D' \cup \{b'\}$ is a continuous bijection, and hence a homeomorphism. Since the family of paths through a given point is of modulus zero by 7.9, f and f^* have the same dilatations (cf. Example 16.2). ∆∆

17.4. THEOREM. R^n cannot be mapped qclly onto a proper subdomain.

Proof. Let $f : R^n \to D' \subset R^n$ be a qc mapping. By 17.3, f has a homeomorphic extension $f^* : \bar{R}^n \to D' \cup \{b'\}$. Then $D' \cup \{b'\}$ is compact and open, which implies $D' \cup \{b'\} = \bar{R}^n$. Thus $b' = \infty$ and $D' = R^n$. ∆∆

We next introduce five concepts which describe the behavior of a domain at a boundary point.

17.5. Definition. Let D be a domain in $\bar{R}^n$ and let $b \in \partial D$.

(1) D is locally connected at b if b has arbitrarily small neighborhoods U such that $U \cap D$ is connected.

(2) D is finitely connected at b if b has arbitrarily small neighborhoods U such that $U \cap D$ has a finite number of components.

(3) D has property P_1 at b if the following condition is

satisfied: If E and F are connected subsets of D such that $b \in \bar{E} \cup \bar{F}$, then $M(\Delta(E, F, D)) = \infty$.

(4) D has property P_2 at b if the following condition is satisfied: For each point $b_1 \in \partial D$, $b_1 \neq b$, there is a compact set $F \subset D$ and a constant $\delta > 0$ such that $M(\Delta(E, F, D)) \geq \delta$ whenever E is a connected set in D such that $\bar{E}$ contains b and b_1.

(5) D is locally quasiconformally collared at b if there is a neighborhood U of b and a homeomorphism g of $U \cap \bar{D}$ onto $\{x \in R^n \mid |x| < 1, x_n \geq 0\}$ such that $g|U \cap D$ is qc. (By Topology, g maps $U \cap \partial D$ onto B^{n-1}.)

(6) D has one of the above properties on the boundary if it has it at every boundary point.

We give some alternate characterizations of some of the above properties.

17.6. THEOREM. D is locally connected at b iff each neighborhood U of b contains a neighborhood V of b such that each pair of points in $V \cap D$ can be joined by a connected set in $U \cap D$. In other words, $V \cap D$ is contained in a component of $U \cap D$.

Proof. The necessity of the above condition is trivial. Conversely, assume that it is satisfied. Let U be a neighborhood of b, and let V be the neighborhood given by the condition. Then $V \cap D$ is contained in a component W of $U \cap D$. Now $V \cup W$ is a neighborhood of b such that $(V \cup W) \cap D = W$ is connected. Hence D is locally connected at b. ΔΔ

17.7. THEOREM. The following conditions are equivalent:

(1) D is finitely connected at b.

(2) Every neighborhood U of b contains a neighborhood V of

b such that $V \cap D$ is contained in the union of a finite number of components of $U \cap D$.

(3) If U is a neighborhood of b and if (x_j) is a sequence of points such that $x_j \to b$ and $x_j \in D$, then there is a subsequence which is contained in a single component of $U \cap D$.

<u>Proof.</u> Suppose first that (2) is satisfied. Let U be a neighborhood of b and let V be the neighborhood given by (2). Then $V \cap D$ is contained in the union of a finite number of components $W_1, \ldots, W_k$ of $U \cap D$. Then $V_1 = V \cup W_1 \cup \ldots \cup W_k$ is a neighborhood of b such that $V_1 \cap D$ has k components $W_1, \ldots, W_k$. Hence D is finitely connected at b.

To prove that (3) implies (2), let U be a neighborhood of b, and set $V_j = \{x \mid q(x, b) < 1/j\}$. If no $V_j \cap D$ is contained in a finite number of components of $U \cap D$, we can find a sequence of points $x_1, x_2, \ldots$ such that $x_j \in V_j \cap D$ and such that the points x_j belong to different components of $U \cap D$. This contradicts (3).

We finally show that (1) implies (3). Let U and (x_j) be as in (3). By (1), there is a neighborhood V of b such that $V \subset U$ and such that $V \cap D$ has a finite number of components. Then there is a subsequence which is contained in a single component of $V \cap D$, and hence in a single component of $U \cap D$. ΔΔ

17.8. <u>Remark.</u> Obviously, the properties (1) and (2) of 17.5 are topological invariants and (3), (4), (5) are qc invariants. More precisely, let D and D' be domains, let $b \in \partial D$, and let $f : \bar{D} \to \bar{D}'$ be a homeomorphism which maps D onto D'. If D is locally connected or finitely connected at b, the same is true for D' at $f(b)$. Moreover, if $f|D$ is qc, then f similarly preserves the properties P_1, P_2, and local qc collaredness.

We next establish some relations between these properties. The following statement is trivial:

17.9. <u>THEOREM.</u> If D is locally connected at b, then D is finitely connected at b. $\triangle\!\triangle$

If D is the disk B^2 minus the radius $\{te_1 \mid 0 \leq t < 1\}$, then D is finitely connected but not locally connected at the points te_1, $0 < t < 1$.

17.10. <u>THEOREM.</u> If D is locally qclly collared at b, D is locally connected at b and has the properties P_1 and P_2 at b.

<u>Proof.</u> Since D is locally qclly collared at b, there is a neighborhood U of b and a homeomorphism $g : U \cap \bar{D} \to B^n_+ \cup B^{n-1}$ such that $g \mid U \cap D$ is qc. Here $B^n_+ = \{x \in B^n \mid x_n > 0\}$. We first show that U can be chosen arbitrarily small. Let V be a neighborhood of b. We can find $r > 0$ such that $r < 1 - |g(b)|$ and such that $|g(x)-g(b)| < r$ implies $x \in V \cap \bar{D}$. Then $U_1 = (V \setminus \bar{D}) \cup g^{-1}B^n(g(b), r)$ is a neighborhood of b. Moreover, $U_1 \subset V$ and $U_1 \cap \bar{D} = g^{-1}B^n(g(b), r)$. Setting $h(y) = (y - g(b))/r$ we obtain a homeomorphism $g_1 = h \circ g : U_1 \cap \bar{D} \to B^n_+ \cup B^{n-1}$ such that g_1 is qc in $U_1 \cap D$. Thus U_1 and g_1 have the same properties as U and g, and $K(g_1) = K(g)$. Moreover, we see that one can choose $g(b) = 0$.

Since B^n_+ is connected, it follows at once that D is locally connected at b. It remains to prove the properties P_1 and P_2.

P_1: Let $g : U \cap \bar{D} \to B^n_+ \cup B^{n-1}$ be as above, with $g(b) = 0$. Let E and F be connected sets in D such that $b \in \bar{E} \cup \bar{F}$. We must prove that $M(\Delta(E, F, D)) = \infty$. Since the modulus is a monotone function and since g is qc, it suffices to prove that $M(\Gamma) = \infty$, where $\Gamma = \Delta(g(E \cap U), g(F \cap U), B^n_+)$. Replacing U by a smaller neighborhood if necessary, we may assume that every hemisphere $S_+(t) = S^{n-1}(t) \cap B^n_+$ meets both $g(E \cap U)$ and $g(F \cap U)$ for $0 < t < 1$. Suppose that $\rho \in F(\Gamma)$, and set $\Gamma(t) = \Delta(g(E \cap U), g(F \cap U), S_+(t))$. Then $\rho | S(t) \in F(\Gamma(t))$ for $0 < t < 1$. By means of 10.2 we obtain

$$\int \rho^n \, dm \geq \int_0^1 dt \int_{S(t)} \rho^n \, dm_{n-1} \geq \int_0^1 M_n^{S(t)}(\Gamma(t)) \, dt \geq b_n \int_0^1 \frac{dt}{t} = \infty .$$

Hence $M(\Gamma) = \infty$.

P_2: Let g and U be as above, and let $b_1 \in \partial D$, $b_1 \neq b$. We may assume that $b_1 \notin U$. Let J be the segment $x_1 = \ldots = x_{n-1} = 0$, $1/4 \leq x_n \leq 1/2$, and set $F = g^{-1}J$. We show that F satisfies the condition in the definition 17.5.(4) of P_2. Let E be a connected set such that $\bar{E}$ contains b and b_1. We must find a positive lower bound for $M(\Delta(E, F, D))$. Since g is qc, it suffices to find a lower bound for $M(\Gamma)$ where $\Gamma = \Delta(g(E \cap U), J, R^n_+)$. If $\rho \in F(\Gamma)$, we again obtain by 10.2,

$$\int \rho^n \, dm \geq \int_{1/4}^{1/2} dt \int_{S(t)} \rho^n \, dm_{n-1} \geq b_n \log 2 .$$

Hence $M(\Gamma) \geq b_n \log 2$. Thus D has property P_2 at b. ∆∆

17.11. <u>Exercise.</u> Prove that D has properties P_1 and P_2 at $b \in \partial D$ if b has a neighborhood U such that $\Lambda_{n-1}(U \cap \partial D) = 0$.
<u>Hint.</u> Prove first, using the idea of Hurewicz-Wallman [1, footnote on p. 104], that 10.2 is true if Γ is replaced by $\Delta(E, F, S \setminus Q)$ where $\Lambda_{n-2}(Q) = 0$. Then apply the same method as above.

17.12. <u>THEOREM.</u> Let D be a domain, let $b \in \partial \bar{D}$, and suppose that b has a neighborhood U such that $U \cap \partial D$ is an (n-1)-dimensional C^1-manifold. Then D is locally qclly collared at b.

<u>Proof.</u> Let T be the tangent hyperplane of ∂D at b, and let

$P : \partial D \to T$ be the orthogonal projection. Then P is a homeomorphism in a ∂D-neighborhood V of b. Denoting by e the unit normal of T, we set $g(x + te) = P(x) + te$, $x \in V$, $t \in R^1$. Then g is a diffeomorphism in a neighborhood of b, and hence qc in a smaller neighborhood of b. In fact, we could use 16.6 to show that $K(g)$ can be made arbitrarily close to 1. We can then find a neighborhood U of b such that g maps $U \cap \bar{D}$ onto a half ball plus its boundary $(n-1)$-ball. Hence D is locally qclly collared at b. The condition $b \in \partial \bar{D}$ was needed to guarantee that D does not lie on both sides of the boundary manifold. ∆∆

We now turn to the boundary extension of qc mappings.

17.13. <u>THEOREM.</u> Suppose that $f : D \to D'$ is a qc mapping and that D has property P_1 at $b \in \partial D$. Then $C(f, b)$ contains at most one point at which D' is finitely connected.

<u>Proof.</u> Suppose that D' is finitely connected at two distinct points b_1', b_2' of $C(f, b)$. Choose ball neighborhoods U_i of b_i' such that $\bar{U}_1 \cap \bar{U}_2 = \emptyset$. There are sequences (x_j) and (y_j) such that $x_j, y_j \in D$, $x_j \to b$, $y_j \to b$, $f(x_j) \to b_1'$ and $f(y_j) \to b_2'$. From 17.7 it follows that $U_1 \cap D'$ has a component E_1 which contains a subsequence of $(f(x_j))$. Similarly, $U_2 \cap D'$ has a component E_2 which contains a subsequence of $(f(y_j))$. Set $\Gamma = \Delta(f^{-1}E_1, f^{-1}E_2, D)$ and $\Gamma' = \Delta(E_1, E_2, D')$. Since $b \in \overline{f^{-1}E_1} \cup \overline{f^{-1}E_2}$, it follows from property P_1 that $M(\Gamma) = \infty$. On the other hand, $\Gamma' > \Gamma_A$ where A is the ring $R(\bar{U}_1, \bar{U}_2)$. Hence $M(\Gamma') \leq M(\Gamma_A) < \infty$ by 11.5. This contradicts the qcty of f. ∆∆

17.14. <u>COROLLARY.</u> If D has property P_1 at $b \in \partial D$, if D' is finitely connected on the boundary and if $f : D \to D'$ is qc, then f has a limit at b . ∆∆

17.15. THEOREM. Suppose that $f : D \to D'$ is a qc mapping and that D is locally connected at $b \in \partial D$. If D' has property P_2 at some point of $C(f, b)$, then f has a limit at b.

Proof. Suppose that $C(f, b)$ contains two distinct points b_1' and b_2' and that D' has property P_2 at b_1'. By the definition of P_2, there is a compact set $F \subset D'$ and a positive number δ such that $M(\Delta(E, F, D')) \geq \delta$ whenever E is a connected set in D' such that $\bar{E}$ contains b_1' and b_2'. Since D is locally connected at b, we can choose a sequence $U_1, U_2, \ldots$ of neighborhoods of b such that every $U_j \cap D$ is connected and such that $q(U_j) \to 0$. Set $\Gamma_j = \Delta(U_j \cap D, f^{-1}F, D)$ and $\Gamma_j' = \Delta(f(U_j \cap D), F, D')$. Since $b_i' \in C(f, b) \subset \overline{f(U_j \cap D)}$, $M(\Gamma_j') \geq \delta$. On the other hand, we can find spherical rings A_j with radii $u_j < v_j$ such that $\Gamma_{A_j} < \Gamma_j$ and $u_j/v_j \to 0$. Consequently, $M(\Gamma_j) \to 0$, which contradicts the qcty of f. ∆∆

One can obtain several results by combining the above theorems. We give some examples. Before that, we give without proof a simple topological result:

17.16. THEOREM. Suppose that $f : D \to D'$ is a homeomorphism such that $\lim_{x \to b} f(x) = f^*(b)$ exists for every b in a set $E \subset \partial D$. Then the extension $f^* : D \cup E \to D' \cup f^*E$ of f is continuous. If, in addition, $\lim_{y \to b'} f^{-1}(y)$ exists for every $b' \in f^*E$, f^* is a homeomorphism. ∆∆

17.17. THEOREM. Suppose that $f : D \to D'$ is a qc mapping, that D is locally qclly collared at every point of $E \subset \partial D$ and that for every $b \in E$, D' is locally qclly collared at some point of $C(f, b)$. Then f can be extended to a homeomorphism $f^*: D \cup E \to D' \cup E'$ where $E' = C(f, E)$.

Proof. If $b \in E$, it follows from 17.10 that D is locally connected at b and that D' has property P_2 at some point of $C(f, b)$. Hence, by 17.15, f has a limit $f^*(b)$ at b. Similarly, f^{-1} has a limit at every point of f^*E. The theorem follows now from 17.16. ∆∆

17.18. THEOREM. Suppose that D_0 is locally qclly collared on the boundary and that D and D' are locally connected on the boundary and qclly equivalent to D_0. Then every qc mapping $f: D \to D'$ can be extended to a homeomorphism $f^*: \bar{D} \to \bar{D}'$.

Proof. We can write $f = h \circ g$ where $g: D \to D_0$ and $h: D_0 \to D'$ are qc mappings. By 17.10 and 17.15, g has a limit at every point of ∂D. By 17.9, 17.10 and 17.14, h has a limit at every point of ∂D_0. Consequently, f has a limit at every point of ∂D. Since the same is true for f^{-1}, the assertion follows from 17.16. ∆∆

17.19. Definition. A domain $D \subset \bar{R}^n$ is a Jordan domain if ∂D is homeomorphic to S^{n-1}.

A Jordan domain need not be homeomorphic to B^n, if $n \geq 3$. A famous counterexample is Alexander's horned sphere (Hocking-Young [1, p. 176]). Even if a Jordan domain is homeomorphic to B^n, it need not be qclly equivalent to B^n. Counterexamples are given in 17.24.3 and in Gehring-Väisälä [2]. It is not known whether a qc mapping between two Jordan domains can always be extended to the boundaries. However, the following weaker result is easily established:

17.20. <u>THEOREM.</u> Let D and D' be Jordan domains which are qclly equivalent to a ball. Then every qc mapping $f : D \to D'$ can be extended to a homeomorphism $f^* : \bar{D} \to \bar{D}'$.

<u>Proof.</u> By a result of Wilder [1, p. 66], every Jordan domain is locally connected on the boundary. Since B^n is locally qclly collared, the theorem follows from 17.18. ∆∆

17.21. <u>Remark.</u> Suppose that D is a simply connected domain in $\bar{R}^2$ and that D is locally connected on the boundary. If ∂D contains more than one point, we can map D conformally onto B^2 by the Riemann mapping theorem. By 17.18, this mapping can be extended to a homeomorphism of $\bar{D}$ onto $\bar{B}^2$. Hence D is a Jordan domain. We have thus given an analytic proof for the following topological theorem: If D is a simply connected domain in $\bar{R}^2$ and if D is locally connected on the boundary, then ∂D is either empty or a point or a Jordan curve. For a topological proof, see Newman [1, p. 167].

17.22. <u>THEOREM.</u> Let D be a domain in $\bar{R}^n$ such that D is locally connected on the boundary but D is not a Jordan domain. Then D is not qclly equivalent to a ball.

<u>Proof.</u> If $f : D \to B^n$ is qc, it follows from 17.18 that f can be extended to a homeomorphism $f^* : \bar{D} \to \bar{B}^n$. Thus D is a Jordan domain. ∆∆

17.23. <u>Examples.</u> From 17.22 it follows that the following domains are not qclly equivalent to a ball:

(1) R^n. This was already proved in 17.4.

(2) A ball minus a radius $(n \geq 3)$.

(3) The complement of a ray $(n \geq 3)$, cf. 16.4.

(4) More generally, the complement of a closed set whose topological dimension is less than $n-1$. (See Hurewicz-Wallman [1, p. 48].)

(5) The domain between two parallel planes ($n=3$).

(6) More generally, the domain $D = B^k \times R^{n-k}$ where $1 \leq k \leq n-2$.

17.24. <u>Remarks.</u> 1. Theorem 17.3 was proved by Loewner [1]. It has the following generalization (Väisälä [4]): If E is closed with respect to a domain D and if $\Lambda_{n-1}(E) = 0$, then every qc mapping f of $D \setminus E$ has a unique extension to a qc mapping f^* of D such that $K_I(f^*) = K_I(f)$ and $K_O(f^*) = K_O(f)$.

2. Theorems 17.20 and 17.22 are from Väisälä [2]. Their proofs were essentially based on the fact that B^n has the properties P_1 and P_2. The definitions of P_1 and P_2 have not been previously published. The property P_1 is due to the author, while the definition of P_2 is due to Näkki who considered it in his unpublished licentiate's thesis. The above results concerning P_2 in 17.10 and 17.15 are also due to him.

(After the manuscript of these notes was completed, Näkki published his doctor's thesis [1], which contains further results in this direction. His terminology is slightly different from ours.)

3. One can easily generalize 17.12 by allowing that the boundary manifold has a corner at b. For example, every polyhedron is locally qclly collared on the boundary. However, 17.12 is not true if we completely drop the differentiability hypothesis on ∂D. In other words, a Jordan domain need not be locally qclly collared on the boundary. As a counterexample consider the domain D in R^3, defined by $x_1 > 0$, $|x_2| < x_1^a$, where $a > 1$. Then D is not locally qclly collared at the points of the edge $x_1 = x_2 = 0$. This can be proved in several ways, see e.g. Gehring-Väisälä [2, p. 62]. In fact, D does not have property P_1, and, if $a \geq 2$, not property P_2 either.

Curiously, it has P_2 if $1 < a < 2$. These results are due to Näkki [1, p. 48].

4. Gehring [6] has proved that if a Jordan domain D in $\bar{R}^3$ is locally qclly collared on the boundary, it is also globally qclly collared in the following sense: There exists a neighborhood U of ∂D and a homeomorphism g of $U \cap \bar{D}$ onto a set $\{x \in R^3 \mid a < |x| \leq 1\}$ such that $g \mid U \cap D$ is qc. It is not known whether the corresponding result is true for $n \geq 4$. Furthermore, Gehring proved that the above global condition implies in all dimensions that D is qclly equivalent to B^n. We shall prove the latter result in Section 41.

5. The Carathéodory theory of prime ends has been generalized to n dimensions by Zorič [1], [2].

18. Distortion

18.1. THEOREM. For every $K \geq 1$ and $n \in N$, $n \geq 2$, there exists a function $\theta_K^n : (0, 1) \to R^1$ with the following properties:

(1) θ_K^n is increasing.

(2) $\lim_{r \to 0} \theta_K^n(r) = 0$.

(3) $\lim_{r \to 1} \theta_K^n(r) = \infty$.

(4) Let D and D' be proper subdomains of R^n and let $f : D \to D'$ be K-qc. If x and y are points in D such that $0 < |y - x| < d(x, \partial D)$, then

$$\frac{|f(y) - f(x)|}{d(f(x), \partial D')} \quad \text{and} \quad \frac{|f(y) - f(x)|}{d(f(y), \partial D')} \quad \text{are} \quad \leq \theta_K^n\left(\frac{|y - x|}{d(x, \partial D)} \right).$$

Proof. Suppose that $f : D \to D'$ and x, y are as in (4). We abbreviate $d = d(x, \partial D)$, $d' = d(f(x), \partial D')$, $d'' = d(f(y), \partial D')$. Let A be the spherical ring $\{z \mid |y-x| < |z-x| < d\}$. Then $A \subset D$. Setting

$C_0 = f\bar{B}^n(x, |y-x|)$ and $C_1 = \complement f B^n(x, d)$, we have $fA = R(C_0, C_1)$. Here C_0 contains $f(x)$ and $f(y)$ while C_1 contains ∞ and points b', b'' in $\partial D'$ such that $|f(x) - b'| = d'$, $|f(y) - b''| = d''$. By 11.9, $M(\Gamma_{fA})$ is not less than the numbers $\varkappa_n(d'/|f(x)-f(y)|)$, $\varkappa_n(d''/|f(x)-f(y)|)$. Since f is K-qc, $M(\Gamma_{fA}) \leq K M(\Gamma_A)$, whence

$$K\omega_{n-1}(\log \frac{d}{|y-x|})^{1-n} \geq \varkappa_n(d'/|f(x)-f(y)|).$$

Set $u_n(t) = \sup\{r \mid \varkappa_n(r) > t\}$. Since $\varkappa_n$ is decreasing, $|f(x) - f(y)|/d' \leq \theta_K^n(|y-x|/d)$ where

$$\theta_K^n(r) = 1/u_n(K\omega_{n-1}(\log \frac{1}{r})^{1-n}).$$

The same inequality holds if d' is replaced by d''. The properties (1), (2), (3) follow from the properties of $\varkappa_n$, given in 11.7. ∆

18.2. <u>THEOREM.</u> For every $K \geq 1$ and $n \in N$, $n \geq 2$, there is a function $\varphi_K^n : [0, 1) \to [0, 1)$ with the following properties:

(1) φ_K^n is increasing.

(2) $\lim_{r\to 0} \varphi_K^n(r) = 0$.

(3) $\lim_{r\to 1} \varphi_K^n(r) = 1$.

(4) If $f : B^n \to B^n$ is a K-qc mapping such that $f(0) = 0$, then $|f(x)| \leq \varphi_K^n(|x|)$ for all $x \in B^n$.

<u>Proof.</u> We apply 18.1 by taking $x = 0$. Since $d(f(y), \partial B^n) = 1 - |f(y)|$, we obtain $|f(y)|/(1 - |f(y)|) \leq \theta_K^n(|y|)$, which implies

$$|f(y)| \leq \frac{\theta_K^n(|y|)}{1 + \theta_K^n(|y|)} = \varphi_K^n(|y|). \quad ∆$$

18.3. <u>Remark.</u> Similar results can be proved for more general domains. In general, a point of a compact set cannot be mapped near a boundary point unless the whole set is mapped near this point. We shall give a precise meaning for this in 21.13.

18.4. Remark. Theorem 18.1 is due to Gehring [3, p. 383]. He also proved that for small r we can choose $\theta_K^n(r) = a_n r^\alpha$ where $\alpha = K^{1/(1-n)}$ and a_n is a constant. Thus a qc mapping is Hölder continuous.

19. Equicontinuity

Suppose that D and D' are proper subdomains of R^n. Let $K \geq 1$, $x_0 \in D$ and $y_0 \in D'$ be fixed, and consider the family of all K-qc mappings $f : D \to D'$ such that $f(x_0) = y_0$. From 18.1 it immediately follows that this family is equicontinuous at x_0. In this section we give some more general results of this type. Since we do not want to exclude the point at infinity, we use the spherical metric.

19.1. Definition. Suppose that T is a topological space, that (M, q) is a metric space and that W is a family of mappings $f : T \to M$. W is equicontinuous at a point $x_0 \in T$ if for each $\varepsilon > 0$ there is a neighborhood U of x_0 such that $q(f(x), f(x_0)) < \varepsilon$ whenever $x \in U$ and $f \in W$. If W is equicontinuous at each point of T, it is called equicontinuous.

In our case, T is always a domain in $\bar{R}^n$, $M = \bar{R}^n$, and q is the spherical metric.

19.2. THEOREM. Suppose that D is a domain in $\bar{R}^n$, that $K \geq 1$ and that $r > 0$. If W is a family of K-qc mappings of D (not necessarily onto a fixed domain) such that each $f \in W$ omits two points a_f, b_f with $q(a_f, b_f) \geq r$, then W is equicontinuous.

Proof. Let $x_0 \in D$ and $0 < \varepsilon < r$. Choose ball neighborhoods U and V of x_0 such that $\bar{U} \subset V \subset D$. Let A be the ring $V \setminus \bar{U}$. If $f \in W$, then $fA = R(C_0, C_1)$ where $C_0 = f\bar{U}$ and $C_1 = \complement fV$. Hence $q(C_1) \geq q(a_f, b_f) \geq r$. If x is an arbitrary point in U, $q(f(x), f(x_0)) \leq q(C_0)$. Using the function $\lambda_n(r)$, defined in 12.4, we thus have $K\,M(\Gamma_A) \geq M(\Gamma_{fA}) \geq \lambda_n(t)$ where $t = \min(r, q(f(x), f(x_0)))$. We choose U so small that $K\,M(\Gamma_A) \leq \lambda_n(\varepsilon)$. Then $\lambda_n(t) \leq \lambda_n(\varepsilon)$ for every $x \in U$ and $f \in W$. Since λ_n is increasing, this implies $t \leq \varepsilon$. Since $\varepsilon < r$, we have $q(f(x), f(x_0)) = t \leq \varepsilon$ whenever $x \in U$ and $f \in W$. Hence W is equicontinuous at x_0. ∆∆

19.3. COROLLARY. If W is a family of K-qc mappings of D such that every $f \in W$ omits two fixed values, then W is equicontinuous. In particular, if D has at least two boundary points, the family of all K-qc mappings of D onto a fixed domain D' is equicontinuous. ∆∆

19.4. THEOREM. Let W be a family of K-qc mappings of D. Then W is equicontinuous if one of the following conditions is satisfied:

(1) There are points $x_1, x_2 \in D$ and a number $r > 0$ such that each $f \in W$ omits a point a_f such that $q(a_f, f(x_i)) \geq r$ for $i = 1, 2$.

(2) There are points $x_1, x_2, x_3 \in D$ and a number $r > 0$ such that each $f \in W$ satisfies the three inequalities $q(f(x_i), f(x_j)) \geq r$, $i \neq j$.

Proof. (1): Set $D_1 = D \setminus \{x_1\}$. Then every $f|D_1$ omits the points a_f and $f(x_1)$. By 19.2, the family of all restrictions $f|D_1$ is equicontinuous. This means that W is equicontinuous at each point of D except possibly at x_1. Considering similarly the restrictions $f \mid D \setminus \{x_2\}$ we conclude that W is equicontinuous also at x_1.

(2): Every $f \mid D \setminus \{x_1, x_2\}$ omits the points $f(x_1)$ and $f(x_2)$. By 19.2, W is equicontinuous at each point of D except possibly at x_1 and x_2. Considering similarly the restrictions $f \mid D \setminus \{x_2, x_3\}$ and $f \mid D \setminus \{x_1, x_3\}$ we conclude that W is equicontinuous also at these points. ΔΔ

19.5. <u>COROLLARY.</u> If W is a family of K-qc mappings of a domain D such that each $f \in W$ assumes at three given points three fixed values, then W is equicontinuous. ΔΔ

19.6. <u>Remark.</u> Suppose that W is a family of K-qc mappings of D and suppose that there are $x_1, x_2 \in D$ and $r > 0$ such that $q(f(x_1), f(x_2)) \geq r$ for all $f \in W$. Then it follows from 19.2 that W is equicontinuous at each point of D, except possibly at x_1 and x_2. However, W need not be equicontinuous at these points. For example, let $D = \bar{R}^n$ and let $f_j(x) = 2^j x$, $f(\infty) = \infty$. Then $W = \{f_j \mid j \in Z\}$ is a family of 1-qc mappings of D, and $q(f_j(0), f_j(\infty)) = 1$ for every integer j. However, f is not equicontinuous at 0 and ∞.

19.7. <u>Remark.</u> This section is an n-dimensional version of Lehto-Virtanen [1, pp. 71-73].

20. Normal families

The purpose of this section is to prove Ascoli's theorem. Let T again be a topological space and let (M, q) be a metric space.

20.1. <u>Definition.</u> A sequence of mappings $f_j : T \to M$ is said to converge <u>c-uniformly</u> to a mapping $f : T \to M$ if $f_j \to f$ uniformly on every compact subset of T.

20.2. <u>Definition.</u> A family W of continuous mappings $f : T \to M$ is called a <u>normal family</u> if every sequence of W has a subsequence which converges c-uniformly in T.

20.3. <u>THEOREM.</u> Let (M, q) be complete, and let $f_j : T \to M$ be an equicontinuous sequence which converges at every point of a set E which is dense in T. Then (f_j) converges c-uniformly in T.

<u>Proof.</u> Let F be a compact set in T, and let $\varepsilon > 0$. From the equicontinuity it follows that every $x \in F$ has a neighborhood $U(x)$ such that $q(f_j(x), f_j(y)) < \varepsilon/5$ whenever $y \in U(x)$ and $j \in N$. We choose a finite covering $\{U(x_1), \dots, U(x_k)\}$ of F. Since E is dense, we can find points $a_i \in U(x_i) \cap E$, $1 \le i \le k$. Since (f_j) converges pointwise in E, there are integers n_i such that $q(f_m(a_i), f_n(a_i)) < \varepsilon/5$ whenever $m \ge n_i$ and $n \ge n_i$, $1 \le i \le k$. Set $n_0 = \max(n_1, \dots, n_k)$. If $x \in F$, $m \ge n_0$ and $n \ge n_0$, then x belongs to some $U(x_i)$, and we obtain $q(f_m(x), f_n(x)) \le q(f_m(x), f_m(x_i)) + q(f_m(x_i), f_m(a_i)) + q(f_m(a_i), f_n(a_i)) + q(f_n(a_i), f_n(x_i)) + q(f_n(x_i), f_n(x)) < \varepsilon$. Since M is complete, (f_j) converges uniformly on F. ∆∆

20.4. <u>ASCOLI'S THEOREM.</u> If T is a separable topological space and if M is a compact metric space, then every equicontinuous family W of mappings $f : T \to M$ is a normal family.

<u>Proof.</u> Let $J = f_1, f_2, \dots$ be a sequence of W. Since T is separable, it contains a countable dense subset $E = \{a_n \mid n \in N\}$.

Since M is compact, J has a subsequence $J_1 = f_{11}, f_{12}, \ldots$ which converges at a_1. By induction, we obtain sequences $J_k = f_{k1}, f_{k2}, \ldots$ such that J_k is a subsequence of J_{k-1} and such that J_k converges at a_k. Then the diagonal sequence $J' = f_{11}, f_{22}, \ldots, f_{kk}, \ldots$ converges at every point of E. By 20.3, J' converges c-uniformly in T. Hence W is a normal family. ΔΔ

By Ascoli's theorem, we obtain the following consequence of the results of Section 19:

20.5. <u>THEOREM.</u> Let D be a domain in $\bar{R}^n$, and let W be a family of K-qc mappings of D such that some of the conditions in 19.2 - 19.5 is valid. Then W is a normal family. ΔΔ

20.6. <u>Remark.</u> The material of this section is classical. Our presentation is from Lehto-Virtanen [1, pp. 74-75]. 20.3 was given as a separate theorem because it is needed in Section 21.

21. Convergent sequences

Suppose that W is a sequence of K-qc mappings of a domain D such that each member of W omits two fixed points. By 19.3 and 20.5, W has a subsequence which converges c-uniformly to a mapping $f : D \to \bar{R}^n$. It is natural to ask what can be said of f. This and related questions will be considered in this section. It turns out that f is either a homeomorphism or a constant. In the first case, f is even K-qc, but this is not proved until in Section 37.

21.1. <u>THEOREM.</u> Let $f_j : D \to D_j$ be a sequence of K-qc mappings which converge pointwise to a mapping $f : D \to \bar{R}^n$. Then there are three possibilities:

A. f is a homeomorphism onto a domain D', and the convergence is c-uniform.

B. f assumes exactly two values, one of which at exactly one point. The convergence is not c-uniform.

C. f is a constant. The convergence may be c-uniform or not.

Proof. Suppose first that f assumes exactly two values $b_1 = f(a_1)$ and $b_2 = f(a_2)$. Then $q(f_j(a_1), f_j(a_2)) \geq r > 0$ for all j. Hence, by Remark 19.6, the family $\{f_j \mid j \in N\}$ is equicontinuous in D except possibly at a_1 and a_2. By 20.3, $f_j \to f$ c-uniformly in $D \setminus \{a_1, a_2\}$. Since $fD = \{b_1, b_2\}$, $f(D \setminus \{a_1, a_2\})$ is either $\{b_1\}$ or $\{b_2\}$. Since f is not continuous, the convergence cannot be c-uniform. We have thus the situation B.

It remains to prove that if f assumes at least three values $b_1 = f(a_1)$, $b_2 = f(a_2)$, $b_3 = f(a_3)$, then we have the situation A. Since $q(f_j(a_i), f_j(a_k)) \geq r > 0$, $\{f_j \mid j \in N\}$ is equicontinuous by 19.4. By 20.3, $f_j \to f$ c-uniformly. Hence f is continuous. By Topology, it suffices to show that f is injective.

Suppose that there are distinct points $z_1, z_2 \in D$ such that $f(z_1) = f(z_2)$. We first prove that every neighborhood U of z_1 contains a point $x_0 \neq z_1$ such that $f(z_1) = f(x_0)$. We choose a sphere $S \subset U$ such that S separates the points z_1, z_2 in $\bar{R}^n$. Then fS separates $f_j(z_1)$ and $f_j(z_2)$ for all j. Consequently, there are points $x_j \in S$ such that

$$(21.2) \qquad q(f_j(x_j), f_j(z_1)) \leq q(f_j(z_2), f_j(z_1)).$$

Passing to a subsequence, we may assume that $x_j \to x_0 \in S$. Since $\{f_j \mid j \in N\}$ is equicontinuous at x_0, $q(f_j(x_j), f(x_0)) \leq q(f_j(x_j), f_j(x_0)) + q(f_j(x_0), f(x_0)) \to 0$ as $j \to \infty$. Hence (21.2) implies $q(f(x_0), f(z_1)) \leq q(f(z_2), f(z_1)) = 0$.

Next we prove that every $x_o \in D$ has a neighborhood U such that $f|U$ is either injective or constant. Since $\{f_j \mid j \in N\}$ is equicontinuous, there is a ball neighborhood U of x_o such that $q(f_j(x), f_j(x_o)) < 1/2$ whenever $x \in U$ and $j \in N$. If U does not have the desired property, we can pick distinct points u_1, u_2, u_3 in U such that $f(u_1) \neq f(u_2) = f(u_3)$. We join u_1 and u_2 by an arc $J_o \subset U$ and choose another arc J_1 such that the end points of J_1 are u_3 and a point $u_4 \in \partial U$ and such that $J_1 \setminus U = \{u_4\}$, $J_o \cap J_1 = \emptyset$. If A is the ring $U \setminus (J_o \cup J_1)$, then $f_j A$ is the ring $R(C_o^j, C_1^j)$ where $C_o^j = f_j J_o$ and $C_1^j = \underline{C} f_j(U \setminus J_1)$. Then $q(C_o^j) \geq q(f_j(u_1), f_j(u_2))$, $q(C_1^j) \geq q(\underline{C} f_j U) = 1$, and $q(C_o^j, C_1^j) \leq q(f_j(u_2), f_j(u_3))$. Using the function $\lambda_n(r, t)$, defined in 12.6, we obtain $M(\Gamma_{A_j}) \geq \lambda_n(r_j, t_j)$ where $r_j = q(f_j(u_1), f_j(u_2)) \to q(f(u_1), f(u_2)) > 0$ and $t_j = q(f_j(u_2), f_j(u_3)) \to q(f(u_2), f(u_3)) = 0$. From 12.7 it follows that $M(\Gamma_{A_j}) \to \infty$. On the other hand, $M(\Gamma_{A_j}) \leq K\, M(\Gamma_A)$. Since A is independent of j, this leads to a contradiction.

To complete the proof, we let D_1 be the set of all $x \in D$ which have a neighborhood in which f is injective, and D_2 the set of all $x \in D$ which have a neighborhood in which f is constant. Then D_1 and D_2 are open and disjoint, and $D = D_1 \cup D_2$. Since $z_1 \notin D_1$, $D_2 \neq \emptyset$. Since D is connected, $D = D_2$. Hence f is constant, which is a contradiction. ∆∆

21.3. <u>COROLLARY.</u> If $f_j : D \to D_j$ is a sequence of K-qc mappings which converge c-uniformly to a mapping $f : D \to \overline{R}^n$, then f is either a homeomorphism onto a domain D' or a constant. ∆∆

21.4. <u>Remark.</u> In Section 37 we shall prove that the homeomorphism in the case A is in fact K-qc.

It is natural to ask how the domain D' in the case A or the constant in the case C is related to the domains D_j. We shall next study these questions in detail. The discussion falls into three parts according as ∂D is empty or a point or a set consisting of at least two points.

21.5. <u>THEOREM.</u> If $f_j : \bar{R}^n \to \bar{R}^n$ is a sequence of K-qc mappings which converge c-uniformly (i.e. uniformly) to a mapping f, then f is a homeomorphism onto $\bar{R}^n$.

<u>Proof.</u> If f is constant, then $q(f_j\bar{R}^n) \to 0$. However, since $f_j\bar{R}^n = \bar{R}^n$, this is impossible. Hence we must have the situation A. ⩠

21.6. <u>Remark.</u> It is possible that f_j converge non-uniformly to a constant. Example: $f_j(x) = x + je_1$, $f_j(\infty) = \infty$.

21.7. <u>THEOREM.</u> Let $D = \bar{R}^n \setminus \{a\}$, and let $f_j : D \to D_j$ be a sequence of K-qc mappings which converge c-uniformly to a mapping f. Then f is either constant or a homeomorphism onto a domain $\bar{R}^n \setminus \{b\}$. In the second case, $b = \lim_{j\to\infty} b_j$ where $D_j = \bar{R}^n \setminus \{b_j\}$.

<u>Proof.</u> Suppose that f is not constant. By 21.3, it is a homeomorphism. By 17.3, f_j can be extended to a K-qc mapping $f_j^* : \bar{R}^n \to \bar{R}^n$, $f_j^*(a) = b_j$. If x_1, x_2, x_3 are distinct points in D, $q(f_j(x_i), f_j(x_k)) \geq r > 0$. Hence $\{f_j^* \mid j \in N\}$ is equicontinuous by 19.4. By 20.3, f_j^* converge uniformly in $\bar{R}^n$ to a mapping f^*. By 21.5, f^* is a homeomorphism onto $\bar{R}^n$. Thus f maps D onto $\bar{R}^n \setminus \{b\}$ where $b = f^*(a) = \lim f_j^*(a) = \lim b_j$. ⩠

We next consider the case where D has at least two boundary points. We need the concept of the kernel of a sequence of sets.

21.8. Definition. Let $E_1, E_2, \ldots$ be a sequence of sets in $\bar{R}^n$. The kernel $\ker_{j\to\infty} E_j$ of this sequence is the set of all points in $\bar{R}^n$ which have a neighborhood which is contained in all but a finite number of the sets E_j. Equivalently,

$$\ker_{j\to\infty} E_j = \bigcup_{k=1}^{\infty} \operatorname{int} \bigcap_{j=k}^{\infty} E_j .$$

The kernel of a sequence is always an open set. However, it need not be connected if the E_j are domains. We shall also use the simpler but less rigorous notation $\ker E_j$ for $\ker_{j\to\infty} E_j$.

21.9. THEOREM. Let D be a domain which has at least two boundary points. Let $f_j : D \to D_j$ be a sequence of K-qc mappings which converge c-uniformly to a mapping f. Then f is either a homeomorphism onto a component of $\ker D_j$ or a constant in $\underline{C}(\ker D_j \cup \ker \underline{C}D_j)$.

Proof. Suppose first that f is a homeomorphism onto D'. We first show that $D' \subset \ker D_j$. Let $y_0 \in D'$, and choose a neighborhood U of $x_0 = f^{-1}(y_0)$ such that $\bar{U} \subset D$. Since $f_j \to f$ uniformly in ∂U, there is a ball neighborhood V of y_0 and an integer j_0 such that $f_j(x_0) \in V$ and $V \cap f_j \partial U = \emptyset$ for $j \geq j_0$. Since V is connected, $V \subset f_j U \subset D_j$ for $j \geq j_0$. Thus $y_0 \in \ker D_j$, whence $D' \subset \ker D_j$.

Let G be the component of $\ker D_j$ which contains D'. Suppose that $D' \neq G$. Then there is a point $b \in G \cap \partial D'$. By the definition of the kernel, there is a ball neighborhood U of b and an integer j_0 such that $U \subset D_j$ for $j \geq j_0$. Hence $g_j = f_j^{-1} | U$ is defined for $j \geq j_0$. Since every g_j omits two fixed values, namely the boundary points of D, $\{g_j \mid j \geq j_0\}$ is a normal family by 19.3 and 20.5. Passing to a subsequence, we may therefore assume that $g_j \to g$ c-uniformly in U. Consider a point $x \in f^{-1}(D' \cap U)$. Since $f_j(x) \to f(x)$ and since $\{g_j \mid j \geq j_0\}$ is equicontinuous at $f(x)$,

$q(g_j(f(x)), x) = q(g_j(f(x)), g_j(f_j(x))) \to 0$. Thus $g(f(x)) = x$ for $x \in f^{-1}(D' \cap U)$. In particular, g is not constant in the non-empty open set $D' \cap U$. By 21.3, g is a homeomorphism of U onto a domain V. By the first part of the proof, $V \subset \ker g_j U \subset D$. Hence $g(b) \in D$. Since $f(g(y)) = y$ for $y \in D' \cap U$, we have $f(g(b)) = \lim f(g(y_i)) = \lim y_i = b$, where $y_1, y_2, \ldots$ is any sequence in $D' \cap U$ converging to b. Thus $b \in D'$, which is a contradiction and proves that $D' = G$.

If f is not a homeomorphism, then, by 21.3, $f(x) = c =$ constant. Since every neighborhood of c meets D_j for large j, $c \in \underline{C}(\ker \underline{C}D_j)$. It remains to show that the assumption $c \in \ker D_j$ leads to a contradiction. Choose a ball neighborhood U of c and an integer j_o such that $U \subset D_j$ for $j \geq j_o$. As above, the family $\{g_j \mid j \geq j_o\}$ is equicontinuous. If $x \in D$, then $f_j(x) \in U$ for large j, and we obtain $q(x, g_j(c)) = q(g_j(f_j(x)), g_j(c)) \to 0$. Hence $g_j(c)$ converges to every point $x \in D$, which gives the contradiction. ∆∆

We next show that in the situation A, the inverse mappings f_j^{-1} converge to f^{-1}.

21.10. <u>THEOREM.</u> Suppose that $f_j : D \to D_j$ is a sequence of K-qc mappings which converge to a homeomorphism $f : D \to D'$. Then for every compact set $F \subset D'$ there is an integer j_o such that $F \subset D_j$ for $j \geq j_o$. Moreover, the mappings $f_j^{-1} \mid F$ converge uniformly to $f^{-1} \mid F$.

<u>Proof.</u> Suppose first that $F \neq D'$. This is always the case if $D' \neq \bar{R}^n$. Choose a domain G such that $F \subset G$ and such that $\bar{G}$ is a proper subset of D'. Since $D' \subset \ker D_j$ by 21.9, we can find for every $y \in \bar{G}$ a neighborhood $U(y)$ and an integer $j(y)$ such that $U(y) \subset D_j$ for $j \geq j(y)$. Choose a finite covering $\{U(y_1), \ldots, U(y_k)\}$ of $\bar{G}$, and set $j_o = \max(j(y_1), \ldots, j(y_k))$. Then $F \subset \bar{G} \subset D_j$ for

$j \geq j_0$, and the first assertion is proved.

The mappings $g_j = f_j^{-1} | G$ are defined for $j \geq j_0$. If a_1, a_2 are two points in $D \setminus f^{-1}\bar{G}$, $f_j(a_i) \notin \bar{G}$ for large j. Hence g_j omits the values a_1, a_2 for large j. By 19.2, $\{g_j \mid j \geq j_0\}$ is equicontinuous. By 20.3, it suffices to prove that $g_j(y_0) \to f^{-1}(y_0)$ for an arbitrary $y_0 \in G$. Choose $\varepsilon > 0$, and set $x_0 = f^{-1}(y_0)$. By equicontinuity, y_0 has a neighborhood $U \subset G$ such that $q(g_j(y), g_j(y_0)) < \varepsilon$ for all $y \in U$ and $j \geq j_0$. Since $f_j(x_0) \to y_0$, there is $j_1 \geq j_0$ such that $f_j(x_0) \in U$ for $j \geq j_1$. For $j \geq j_1$ we then have $q(x_0, g_j(y_0)) = q(g_j(f_j(x_0)), g_j(y_0)) < \varepsilon$. Hence $g_j(y_0) \to x_0$.

Finally, if $F = D' = \bar{R}^n$, we can choose compact proper subsets $F_1, F_2 \subset D'$ such that $F_1 \cup F_2 = F$. By what was proved above, $f_j^{-1} \to f^{-1}$ uniformly in each F_i, and hence uniformly in F. ΔΔ

We next consider the special case where D_j is independent of j.

21.11. <u>THEOREM.</u> Suppose that D is a domain which has at least two boundary points and that $f_j : D \to D'$ is a sequence of K-qc mappings onto a fixed domain D' such that $f_j \to f$ pointwise in D. Then D' has at least two boundary points, and the convergence is c-uniform. The limit mapping f is either a homeomorphism onto D' or a constant $c \in \partial D'$. In the first case, $f_j^{-1} \to f^{-1}$ c-uniformly in D'. The second case can occur only in the following cases: (1) ∂D is connected. (2) ∂D consists of two points. (3) ∂D has an infinite number of components.

<u>Proof.</u> From 17.3 it follows that $\partial D'$ has at least two points. By 19.3, $\{f_j \mid j \in N\}$ is equicontinuous. By 20.3, $f_j \to f$ c-uniformly. By 21.9, f is either a homeomorphism onto D' or a constant

$c \in \partial D'$. In the first case, $f_j^{-1} \to f^{-1}$ c-uniformly by 21.10. Suppose next that $f(x) = c =$ constant and that ∂D has exactly k components $B_1, \ldots, B_k$, where $2 \leq k < \infty$. We must show that $k = 2$ and that each B_i consists of only one point. By 17.2, $\partial D'$ has exactly k components $B_1', \ldots, B_k'$ such that for each j, B_i' is one of the cluster sets $C(f_j, B_m)$. Passing to a subsequence, we may assume that $B_i' = C(f_j, B_i)$ for all $j \in N$ and $1 \leq i \leq k$. Choose a compact set $F \subset D$ such that the sets B_i are contained in different components of $\underline{C}F$. For example, we may put $F = \{x \in D \mid q(x, \partial D) \geq r\}$ for a sufficiently small r. Then also the sets B_i' are contained in different components of $\underline{C}f_jF$. Since $f_j(x) \to c$ uniformly in F, $q(f_jF) \to 0$. Denoting by U_j the component of $\underline{C}f_jF$ which has the largest spherical diameter, we have $q(\underline{C}U_j) = q(f_jF)$ for large j, whence $q(\underline{C}U_j) \to 0$. On the other hand, $\underline{C}U_j$ contains all but one B_i'. This is possible only if $k = 2$ and if one of the sets B_i', say B_2', contains only the point c. By 17.3, f_j can be extended to a K-qc mapping $f_j^* : D \cup B_2 \to D' \cup B_2'$. Then $f_j^*(x) \to c$ pointwise in $D \cup B_2$. If B_1 contains more than one point, $\{f_j^* \mid j \in N\}$ is equicontinuous by 19.3. Hence, by 20.3, $f_j^* \to c$ c-uniformly. By what was proved above, c is a boundary point of $D' \cup B_2'$. This contradiction proves that B_1 contains only one point. $\Delta\!\Delta$

21.12. <u>Remark.</u> The limit function can actually be constant in the three cases mentioned in 21.11. As examples we may consider the mappings $f_j(x) = x + je_1$, and set $D = D' = \{x \mid x_n > 0\}$ for the case (1), and $D = D' = \underline{C}(\{ie_1 \mid i \in Z\} \cup \{\infty\})$ for the case (3). For the case (2), let $D = D' = \underline{C}\{0, \infty\}$, and set $f_j(x) = jx$.

We give an application which is closely related to the results of Section 18. In particular, 18.2 is an easy corollary of it.

21.13. THEOREM. Suppose that D and D' are domains, each of which has at least two boundary points. Suppose also that F is a compact set in D and that $K \geq 1$. Then for every $\varepsilon > 0$ there is $\delta > 0$ with the following property: If $f : D \to D'$ is a K-qc mapping such that $q(fF, \partial D') < \delta$, then $q(fF) < \varepsilon$.

Proof. Suppose that the theorem is not true. Then there is $\varepsilon > 0$ and a sequence of K-qc mappings $f_j : D \to D'$ such that $q(f_j F, \partial D') < 1/j$ and $q(f_j F) \geq \varepsilon$. Since every f_j omits two fixed values, $\{f_j \mid j \in N\}$ is a normal family. We may therefore assume that $f_j \to f$ c-uniformly in D. By 21.11, f is either a homeomorphism onto D' or a constant $c \in \partial D'$. The first case is impossible, because $q(f_j F, \partial D') \to 0$. The second case is impossible, because $q(f_j F) \geq \varepsilon$.
ΔΔ

Making use of the last statement of Theorem 21.11, we similarly obtain the following result:

21.14. THEOREM. Suppose that D and D' are domains such that ∂D has at least three points and exactly k components, $2 \leq k < \infty$. Suppose also that F is a compact set in D, consisting of at least two points, and that $K \geq 1$. Then there exists a positive number δ such that $q(fF, \partial D') > \delta$ and $q(fF) > \delta$ for every K-qc mapping $f : D \to D'$. ΔΔ

21.15. Remark. This section is a slightly enlarged n-dimensional version of Lehto-Virtanen [1, pp. 76-82]. See also Gehring [5].

22. The linear dilatation

In this section we define the linear dilatation of a homeomorphism and show that it is bounded if the mapping is qc.

Consider a homeomorphism $f : D \to D'$. Suppose that $x \in D$, $x \neq \infty$, $f(x) \neq \infty$. For each $r > 0$ such that $S^{n-1}(x, r) \subset D$ we set

$$(22.1) \qquad \begin{aligned} L(x, f, r) &= \max_{|y-x|=r} |f(y) - f(x)|, \\ \ell(x, f, r) &= \min_{|y-x|=r} |f(y) - f(x)|. \end{aligned}$$

22.2. Definition. The linear dilatation of f at x is the number

$$H(x, f) = \limsup_{r \to 0} \frac{L(x, f, r)}{\ell(x, f, r)}.$$

If $x = \infty$, $f(x) \neq \infty$, we define $H(x, f) = H(0, f \circ u)$ where u is the inversion $u(x) = x/|x|^2$. If $f(x) = \infty$, we define $H(x, f) = H(x, u \circ f)$.

Obviously, $1 \leq H(x, f) \leq \infty$. If $A : R^n \to R^n$ is a bijective linear mapping, $H(x, A) = H(A)$ for all $x \in R^n$, where $H(A)$ is defined in 14.1. If f is differentiable at x and if $J(x, f) \neq 0$, then $H(x, f) = H(f'(x))$. If f is a K-qc diffeomorphism, (14.3) and 15.1 imply that $H(x, f) \leq K^{2/n}$ for all $x \in D$. This inequality need not be true if f is a K-qc mapping which is not differentiable at x. However, we show that $H(x, f)$ is bounded if f is qc. For later purposes, we formulate the result as follows:

22.3. THEOREM. Suppose that $f : D \to D'$ is a homeomorphism such that one of the following conditions is satisfied for some finite K:

(1) $M(\Gamma_A) \leq K M(\Gamma'_A)$ for all rings A such that $\bar{A} \subset D$.

(2) $K_O(f) \leq K$.

(3) $K_I(f) \leq K$.

Then $H(x, f)$ is bounded by a constant which depends only on n and K.

<u>Proof.</u> Let $x_o \in D$, and set $y_o = f(x_o)$. Performing preliminary inversions if necessary we may assume that $x_o \neq \infty \neq y_o$. Choose $r > 0$ such that $\bar{B}^n(x_o, r) \subset D \setminus \{f^{-1}(\infty)\}$ and $\bar{B}^n(y_o, L(x_o, f, r)) \subset D' \setminus \{f(\infty)\}$. We abbreviate $L = L(x_o, f, r)$, $\ell = \ell(x_o, f, r)$, $L' = L(y_o, f^{-1}, L)$, $\ell' = \ell(y_o, f^{-1}, L)$. Clearly $\ell' = r$. If $\ell < L$, we let A' be the spherical ring $B^n(y_o, L) \setminus \bar{B}^n(y_o, \ell)$ and set $A = f^{-1}A'$.

Suppose that (1) holds. We write $A = R(C_o, C_1)$ where $C_o = f^{-1}\bar{B}^n(y_o, \ell)$ and $C_1 = \complement f^{-1}B^n(y_o, L)$. Then $x_o \in C_o$, $\infty \in C_1$, and the sphere $S^{n-1}(x_o, r)$ meets both C_o and C_1. Hence, by 11.9, $M(\Gamma_A) \geq \varkappa_n(1)$. On the other hand, $M(\Gamma_A) \leq K M(\Gamma'_A) = K\omega_{n-1}(\log\frac{L}{\ell})^{1-n}$. This gives an upper bound for L/ℓ which is trivially true also if $\ell = L$. Letting $r \to 0$ we obtain

$$H(x_o, f) \leq \exp((K\omega_{n-1}/\varkappa_n(1))^{\frac{1}{n-1}}).$$

Since (2) implies (1), it remains to prove the case (3). Choose x_1 such that $|x_1 - x_o| = r$ and $|f(x_1) - f(x_o)| = \ell$. Set $E = A \cap \{x_o + t(x_1 - x_o) \mid t \geq 1\}$, $F = A \cap \{x_o + t(x_1 - x_o) \mid t \leq 0\}$, and $\Gamma = \Delta(E, F, A)$. Since $d(E, F) \geq r$, 7.1 yields $M(\Gamma) \leq m(A)/r^n \leq \Omega_n L'^n/\ell'^n$. On the other hand, 10.12 implies $M(\Gamma') \geq c_n \log\frac{L}{\ell}$. Since $M(\Gamma') \leq K M(\Gamma)$, we obtain $L/\ell \leq \exp(K\Omega_n L'^n/c_n \ell'^n)$, which is trivially true if $\ell = L$. Letting $r \to 0$ yields

$$H(x_o, f) \leq \exp(K\Omega_n H(y_o, f^{-1})^n/c_n).$$

Since $K_O(f^{-1}) \leq K$, the part (2) of the theorem implies that $H(y, f^{-1})$ is bounded by a constant which depends only on n and K.

Thus $H(x, f)$ is also bounded by such a constant. $\Delta\Delta$

22.4. COROLLARY. If $f : D \to D'$ is qc, then $H(x, f)$ is bounded. $\Delta\Delta$

22.5. Remarks. 1. In Section 34 we shall prove the converse: If $H(x, f)$ is bounded, then f is qc. Thus the linear dilatation can be used to define qcty. Moreover, the properties $K_I(f) < \infty$ and $K_O(f) < \infty$ are equivalent.

2. This section is from Väisälä [1]. A slightly different treatment has been given by Gehring [3].

CHAPTER 3. BACKGROUND IN REAL ANALYSIS

In the first two chapters, we have used only rather modest tools of real analysis. However, for a deeper study of qc mappings one must make use of more advanced methods. These will be presented in this chapter. Proofs of the best known facts are omitted, especially in Section 23. The chapter consists of sections 23-30.

23. Set functions

Since all set functions we need are non-negative, locally finite and completely additive, we shall use the term "set function" in the following restricted sense:

23.1. Definition. Let U be an open set in R^n. A set function in U is a function which associates to every Borel set $A \subset U$ a number $\varphi(A) \in \dot{R}^1$ such that the following conditions are satisfied:

(1) $\varphi(A) \geq 0$ for all A.

(2) $\varphi(A) < \infty$ if A is compact.

(3) If $A_1, A_2, \ldots$ is a sequence of disjoint Borel sets in U, then $\varphi(\cup A_i) = \sum \varphi(A_i)$.

A set function φ is absolutely continuous if for every $\varepsilon > 0$ there is $\delta > 0$ such that $m(A) < \delta$ implies $\varphi(A) < \varepsilon$.

23.2. THEOREM. Let φ be a set function which assumes only finite values. Then φ is absolutely continuous iff $\varphi(A) = 0$ whenever A is a Borel set such that $m(A) = 0$. $\Delta\!\!\Delta$

23.3. Definition. Let A be a bounded measurable set in R^n. The parameter of regularity of A is the number

$$r(A) = \frac{m(A)}{\inf m(Q)} ,$$

where the infimum is taken over all cubes $Q \supset A$. By a cube we mean a set $\{x \mid |x_i - a_i| \leq h\}$.

For example, if A is a ball, $r(A) = \Omega_n/2^n$.

23.4. Definition. A set function φ is U is said to have a derivative $\varphi'(x)$ at $x \in U$ if

$$\varphi'(x) = \lim_{j\to\infty} \frac{\varphi(A_j)}{m(A_j)}$$

whenever (A_j) is a sequence of closed sets such that $x \in A_j \subset U$, $d(A_j) \to 0$, $m(A_j) > 0$, and $\inf_j r(A_j) > 0$.

23.5. LEBESGUE'S THEOREM. Let φ be a set function in U. Then

(1) φ has a finite derivative $\varphi'(x)$ a.e.

(2) φ' is measurable in U.

(3) If A is a Borel set in U, $\varphi(A) \geq \int_A \varphi' \, dm$.

(4) If φ is absolutely continuous, (3) holds with equality. $\Delta\!\Delta$

23.6. THEOREM. Suppose that U is an open set in R^n and that $f : U \to \dot{R}^1$ is a non-negative locally integrable function. Then

$$\varphi(A) = \int_A f \, dm$$

defines a set function φ such that $\varphi'(x) = f(x)$ a.e.

Proof. It suffices to show that $\int_A f \, dm = \int_A \varphi' \, dm$ for every compact $A \subset U$. Choose an open set V such that $A \subset V$ and such that

$\bar{V}$ is a compact subset of U. By 23.2, φ is absolutely continuous. The assertion follows from 23.5.(4). ∆∆

23.7. <u>Definition.</u> Let E be a measurable set in R^n. A point $x \in R^n$ is called a <u>point of density</u> of E if $\varphi_E'(x) = 1$ where φ_E is the set function defined by $\varphi_E(A) = m(E \cap A)$.

23.8. <u>DENSITY THEOREM.</u> Almost every point of a measurable set E is a point of density of E.

<u>Proof.</u> Let f be the characteristic function of E. Then $\varphi_E(A) = \int_A f\,dm$. By 23.6, $\varphi_E'(x) = f(x) = 1$ a.e. in E. ∆∆

24. The volume derivative

Suppose that D and D' are domains in R^n and that $f : D \to D'$ is a homeomorphism. Then f maps every Borel set $A \subset D$ onto a Borel set. Denoting $\mu_f(A) = m(fA)$ we clearly obtain a set function μ_f in D. By Lebesgue's theorem 23.5, the derivative $\mu_f'(x)$ exists a.e. in D.

24.1. <u>Definition.</u> $\mu_f'(x)$ is the <u>volume derivative</u> of f at x. Thus

$$\mu_f'(x) = \lim_{r \to 0} \frac{m(f\bar{B}^n(x, r))}{\Omega_n r^n} .$$

From Lebesgue's theorem we obtain:

24.2. <u>THEOREM.</u> (1) $\mu_f'(x) < \infty$ a.e.

(2) μ_f' is measurable.

(3) If A is a Borel set in D, $m(fA) \geq \int_A \mu_f' \, dm$. ΔΔ

24.3. Remark. We excluded the point at infinity in order that μ_f be locally finite. However, the above discussion applies also to the case where $D, D' \subset \bar{R}^n$, since we may consider the restriction of f to $D \setminus \{\infty, f^{-1}(\infty)\}$. In particular, 24.2 is true also in this general case.

24.4. THEOREM. If f is differentiable at x, $\mu_f'(x) = |J(x, f)|$.

ΔΔ

24.5. THEOREM. Suppose that $f : D \to D'$ is a homeomorphism and that $g : D' \to \dot{R}^1$ is a non-negative Borel function. Then

$$\int_{D'} g \, dm \geq \int_D g(f(x)) \, \mu_f'(x) \, dm(x) ,$$

where we use the agreement $\infty \cdot 0 = 0 \cdot \infty = 0$.

Proof. We may assume that $\int_{D'} g \, dm < \infty$. Choose $\varepsilon > 0$, set $h = g \circ f$, and consider the sets

$$A_k = \{x \in D \mid (1+\varepsilon)^k \leq h(x) < (1+\varepsilon)^{k+1}\} ,$$

$$A_\infty = \{x \in D \mid h(x) = \infty\} ,$$

$$A_{-\infty} = \{x \in D \mid h(x) = 0\} .$$

Since $g(y) = \infty$ if $y \in fA_\infty$, $m(fA_\infty) = 0$. By 24.2 this implies $\int_{A_\infty} \mu_f' \, dm = 0$, that is, $\mu_f'(x) = 0$ a.e. in A_∞. Consequently,

$$\int_{A_\infty} h\mu_f' \, dm = 0 .$$

Since also

$$\int_{A_{-\infty}} h\mu_f' \, dm = 0 ,$$

we obtain

$$\int_D h\mu_f' \, dm = \sum_{k\in Z} \int_{A_k} h\mu_f' \, dm \le \sum_{k\in Z} (1+\varepsilon)^{k+1} \int_{A_k} \mu_f' \, dm .$$

Applying 24.2 we further obtain

$$\int_D h\mu_f' \, dm \le (1+\varepsilon) \sum_{k\in Z} (1+\varepsilon)^k m(fA_k) \le (1+\varepsilon) \sum_{k\in Z} \int_{fA_k} g \, dm$$

$$\le (1+\varepsilon) \int_{D'} g \, dm .$$

Since ε was arbitrary, this proves the theorem. ∆∆

24.6. Definition. A homeomorphism $f : D \to D'$ satisfies the condition (N) if $m(A) = 0$ implies $m(fA) = 0$.

24.7. THEOREM. A homeomorphism $f : D \to D'$ satisfies the condition (N) iff $m(fF) = 0$ whenever $F \subset D$ is a compact set of measure zero.

Proof. Suppose that $A \subset D$ and $m(A) = 0$. Then there is a Borel set $E \supset A$ such that $m(E) = 0$. If $m(fE) > 0$, there is a compact set $F' \subset fE$ such that $m(F') > 0$. On the other hand, $m(f^{-1}F') \le m(E) = 0$. Thus the condition of the theorem implies the condition (N). The converse is trivial. ∆∆

24.8. THEOREM. Suppose that $f : D \to D'$ is a homeomorphism which satisfies the condition (N). Then f maps every measurable set $A \subset D$ onto a measurable set, and

$$m(fA) = \int_A \mu_f' \, dm . \tag{24.9}$$

If, in addition, f^{-1} satisfies the condition (N), $\mu_f'(x) > 0$ a.e.

Proof. Suppose first that A is a Borel set in D and that

$m(fA) < \infty$. Choose an open set U such that $A \subset U \subset D$ and $m(fU) < \infty$. By 23.2, μ_f is absolutely continuous in U. By Lebesgue's theorem 23.5, this implies (24.9).

Next let $A \subset D$ be a Borel set with $m(fA) = \infty$. For $r > 0$ choose a Borel set $A_r \subset A$ such that $r < m(fA_r) < \infty$. Then

$$r < m(fA_r) = \int_{A_r} \mu_f' \, dm \leq \int_A \mu_f' \, dm ,$$

which proves that $\int_A \mu_f' \, dm = \infty$. Thus (24.9) holds for every Borel set A.

Next suppose that $A \subset D$ is measurable. Choose a Borel set $E \subset A$ such that $m(A \setminus E) = 0$. Since $m(f(A \setminus E)) = 0$, fA is measurable. Moreover,

$$\int_A \mu_f' \, dm = \int_E \mu_f' \, dm = m(fE) = m(fA) .$$

Thus (24.9) holds for every measurable A.

Finally, let $T = \{x \in D \mid \mu_f'(x) = 0\}$. By (24.9),

$$m(fT) = \int_T \mu_f' \, dm = 0 .$$

If f^{-1} satisfies the condition (N), this implies $m(T) = 0$. ∆∆

25. Partial derivatives

Suppose that U is an open set in R^n and that $f : U \to R^m$ is a mapping. If the i^{th} partial derivative of f exists at a point $x \in U$, we denote it by $\partial_i f(x)$. That is,

$$\partial_i f(x) = \lim_{r \to 0} \frac{f(x+re_i) - f(x)}{r} .$$

If f is differentiable at x, then all partial derivatives exist, and $\partial_i f(x) = f'(x)e_i$. For an arbitrary mapping $f : U \to R^1$ we define the partial Dini derivatives as follows:

$$\partial_i^+ f(x) = \limsup_{r\to 0+} \frac{f(x+re_i) - f(x)}{r},$$

$$\partial_{i+} f(x) = \liminf_{r\to 0+} \frac{f(x+re_i) - f(x)}{r},$$

$$\partial_i^- f(x) = \limsup_{r\to 0-} \frac{f(x+re_i) - f(x)}{r},$$

$$\partial_{i-} f(x) = \liminf_{r\to 0-} \frac{f(x+re_i) - f(x)}{r}.$$

Thus $\partial_i f(x)$ exists iff all these four derivatives are finite and equal.

25.1. <u>THEOREM.</u> The partial Dini derivatives of a continuous function $f : U \to R^1$ are Borel functions.

<u>Proof.</u> We show that $\partial_i^+ f$ is a Borel function. Let $a \in R^1$, and set $E = \{x \in U \mid \partial_i^+ f(x) < a\}$. Then $E = \cup E_j$, where E_j is the set of all $x \in U$ such that $(f(x+re_i) - f(x))/r \leq a - 1/j$ whenever $x+re_i \in U$ and $0 < r < 1/j$. Since f is continuous, each E_j is closed in U. Hence E is a Borel set. ΔΔ

25.2. <u>THEOREM.</u> Let $f : U \to R^m$ be continuous, and let A_i be the set of all $x \in U$ such that $\partial_i f(x)$ exists. Then A_i is a Borel set, and $\partial_i f$ is a Borel function in A_i.

<u>Proof.</u> We write $f(x) = \sum_{j=1}^{m} f_j(x) e_j$. Then each $f_j : U \to R^1$ is continuous. If A_{ij} is the set of all $x \in U$ such that $\partial_i f_j(x)$ exists, $A_i = A_{i1} \cap \ldots \cap A_{im}$. On the other hand, $x \in A_{ij}$ iff $\partial_i^+ f_j(x) = \partial_{i+} f_j(x) = \partial_i^- f_j(x) = \partial_{i-} f_j(x) < \infty$. The theorem follows now from 25.1. ΔΔ

26. ACL-mappings

26.1. Notation. We denote $R_i^{n-1} = \{x \in R^n \mid x_i = 0\}$. Furthermore, P_i is the orthogonal projection of R^n onto R_i^{n-1}. Explicitly, $P_i x = x - x_i e_i$.

26.2. Definition. Let $Q = \{x \in R^n \mid a_i \le x_i \le b_i\}$ be a closed n-interval. A mapping $f : Q \to R^m$ is said to be ACL (absolutely continuous on lines) if f is continuous and if f is absolutely continuous on almost every line segment in Q, parallel to the coordinate axes. More precisely, if E_i is the set of all $x \in P_i Q$ such that the mapping $t \mapsto f(x+te_i)$ is not absolutely continuous on $[a_i, b_i]$, then $m_{n-1}(E_i) = 0$ for $1 \le i \le n$.

If U is an open set in R^n, a mapping $f : U \to R^m$ is called ACL if $f \mid Q$ is ACL for every closed interval $Q \subset U$.

If D and D' are domains in $\bar{R}^n$, a homeomorphism $f : D \to D'$ is called ACL if $f \mid D \setminus \{\infty, f^{-1}(\infty)\}$ is ACL.

We omit the easy proof of the following result:

26.3. THEOREM. Suppose that f is a mapping of an open set $U \subset R^n$ into R^m. If every point x in U has a neighborhood $V(x)$ such that $f \mid V(x)$ is ACL, then f is ACL. ∆∆

26.4. THEOREM. If $f : U \to R^m$ is ACL, the partial derivatives of f exist a.e. in U, and they are Borel functions.

Proof. Fix i, and let E be the set of all $x \in U$ such that $\partial_i f(x)$ does not exist. It suffices to show that $m(E \cap Q) = 0$ for every closed n-interval $Q \subset U$. Since f is continuous, it follows from 25.2 that E is a Borel set. We can thus apply Fubini's theorem which yields

$$m(E \cap Q) = \int_{P_i Q} m_1(P_i^{-1}(x) \cap E \cap Q)\, dm_{n-1}(x) .$$

If f is absolutely continuous on the segment $P_i^{-1}(x) \cap Q$, then $\partial_i f(x)$ exists a.e. on this segment, that is, $m_1(P_i^{-1}(x) \cap E \cap Q) = 0$. Since f is ACL, this implies $m(E \cap Q) = 0$. Finally, $\partial_i f$ is a Borel function by 25.2. ∆

26.5. Definition. An ACL-mapping $f : U \to R^m$ is said to be ACL^p, $p \geq 1$, if the partial derivatives of f are locally L^p-integrable. A homeomorphism $f : D \to D'$ is ACL^p if the restriction of f to $D \setminus \{\infty, f^{-1}(\infty)\}$ is ACL^p.

26.6. Remark. For convenience, we have restricted ourselves to continuous mappings. In the literature, one often assumes only that f is locally or globally L^1- or L^p-integrable, and the class of ACL^p-mappings is denoted by W_p^1 or by H_p^1.

27. Smoothing of functions

Throughout this section we assume that $f : R^n \to R^1$ is a given locally integrable function. Furthermore, $\varphi : R^n \to R^1$ is a continuous mapping with compact support. The support $\operatorname{spt}\varphi$ of φ is the closure of $\{x \mid \varphi(x) \neq 0\}$. The convolution $f * \varphi$ is defined by

$$f * \varphi(x) = \int_{R^n} f(x-y)\, \varphi(y)\, dm(y) = \int_{R^n} f(y)\, \varphi(x-y)\, dm(y) .$$

Thus $f*\varphi$ is a mapping $R^n \to R^1$. It is well defined, because setting $A = \max |\varphi|$ we obtain

$$\int_{R^n} |f(x-y)\,\varphi(y)|\,dm(y) = \int_{\operatorname{spt}\varphi} |f(x-y)\,\varphi(y)|\,dm(y) \le A\int_{\operatorname{spt}\varphi} |f(x-y)|\,dm(y)$$

$$= A\int_{x-\operatorname{spt}\varphi} |f(y)|\,dm(y) < \infty .$$

27.1. <u>THEOREM.</u> $f*\varphi$ is continuous.

<u>Proof.</u> Since φ is uniformly continuous, $|\varphi(x)-\varphi(y)| \le \varepsilon(|x-y|)$ where $\varepsilon(r) \to 0$ as $r \to 0$. Assuming that $|x-x_0| < 1$ we obtain

$$|f*\varphi(x) - f*\varphi(x_0)| \le \int |f(y)|\,|\varphi(x-y) - \varphi(x_0-y)|\,dm(y)$$

$$\le \varepsilon(|x-x_0|)\int_E |f(y)|\,dm(y)$$

where $E = x_0 + \bar{B}^n - \operatorname{spt}\varphi$. ∆∆

27.2. <u>THEOREM.</u> If $\varphi \in C^1$, then $f*\varphi \in C^1$ and $\partial_i(f*\varphi) = f*\partial_i\varphi$.

<u>Proof.</u> Using the mean value theorem we obtain

$$\left| \frac{f*\varphi(x+re_i) - f*\varphi(x)}{r} - f*\partial_i\varphi(x) \right| \le$$

$$\int |f(y)| \left| \frac{\varphi(x+re_i-y) - \varphi(x-y)}{r} - \partial_i\varphi(x-y) \right| dm(y) =$$

$$\int |f(y)|\,|\partial_i\varphi(x_y-y) - \partial_i\varphi(x-y)|\,dm(y)$$

where $|x_y - x| \le r$. The assertion follows as in the proof of 27.1. ∆∆

27.3. <u>THEOREM.</u> If $f \in L^p$, then

$$\lim_{y\to 0} \int |f(x+y) - f(x)|^p\,dm(x) = 0 .$$

<u>Proof.</u> Let $\varepsilon > 0$. Choose $a > 1$ such that

$$\int_{|x|\geq a-1} |f(x)|^p \, dm(x) < \varepsilon .$$

Set $I_y(A) = \int_A |f(x+y) - f(x)|^p \, dm(x)$ and assume that $|y| \leq 1$. Making use of the general inequality $(b+c)^p \leq 2^p(b^p + c^p)$, valid for $b \geq 0$, $c \geq 0$, we obtain

$$I_y(\complement B^n(a)) \leq 2^p \int_{|x|\geq a} (|f(x+y)|^p + |f(x)|^p) \, dm(x) \leq 2^{p+1} \varepsilon .$$

By 23.2, the set function $A \mapsto \int_A |f|^p \, dm$ is absolutely continuous. Hence there is $r > 0$ such that $m(A) < r$ implies $\int_A |f|^p \, dm < \varepsilon$. Thus $I_y(A) < 2^{p+1} \varepsilon$ whenever $m(A) < r$. Applying Lusin's theorem to $f | B^n(a+1)$ we find a compact set $F \subset B^n(a+1)$ such that $f|F$ is continuous and $m(B^n(a+1) \setminus F) < r$. Since F is compact, $f|F$ is uniformly continuous. Consequently, there is $\delta \in (0,1)$ such that $|f(x+y) - f(x)|^p < \varepsilon / m(F)$ whenever $x \in F$, $x+y \in F$, and $|y| < \delta$.

Assume now that $|y| < \delta$. Combining the above inequalities we obtain

$$\int |f(x+y) - f(x)|^p \, dm(x) \leq \int_{\substack{x\in F \\ x+y\in F}} + \int_{\substack{|x+y|<a+1 \\ x+y\notin F}} + \int_{\substack{|x|<a+1 \\ x\notin F}} + \int_{|x|\geq a}$$

$$\leq \varepsilon + 2^{p+1}\varepsilon + 2^{p+1}\varepsilon + 2^{p+1}\varepsilon = (1 + 3\cdot 2^{p+1})\,\varepsilon . \quad \Delta$$

27.4. We now choose a sequence of mappings $\varphi_j : R^n \to R^1$ such that the following conditions are satisfied:

(1) $\varphi_j \in C^1$,

(2) $\varphi_j \geq 0$,

(3) $\operatorname{spt} \varphi \subset B^n(1/j)$,

(4) $\int \varphi_j \, dm = 1$.

For example, after choosing φ_1, we may put $\varphi_j(x) = j^n \varphi_1(jx)$. This sequence will be kept fixed for the rest of this section.

27.5. <u>THEOREM.</u> If $f \in L^p$, then $f*\varphi_j \to f$ in L^p.

<u>Proof.</u> Set $\Delta_j = \int |f*\varphi_j - f|^p \, dm$. We must show that $\Delta_j \to 0$ as $j \to \infty$. We first estimate the integrand by means of Hölder's inequality.

$$\begin{aligned} |f*\varphi_j(x) - f(x)|^p &= \left|\int (f(x-y)\varphi_j(y) - f(x)\varphi_j(y))\, dm(y)\right|^p \\ &\leq \left(\int |f(x-y) - f(x)|\varphi_j(y)\, dm(y)\right)^p \\ &\leq \int |f(x-y) - f(x)|^p \varphi_j(y)\, dm(y) \left(\int \varphi_j \, dm\right)^{p-1} \\ &= \int |f(x-y) - f(x)|^p \varphi_j(y)\, dm(y) . \end{aligned}$$

Integrating over $x \in R^n$ and applying Fubini's theorem (in which we need the fact that $(x, y) \mapsto |f(x-y) - f(x)|^p \varphi_j(y)$ is a measurable function in $R^n \times R^n$) we obtain

$$\begin{aligned} \Delta_j &\leq \int dm(x) \int |f(x-y) - f(x)|^p \varphi_j(y)\, dm(y) \\ &\leq \int_{|y|<1/j} \varphi_j(y)\, dm(y) \int |f(x-y) - f(x)|^p \, dm(x) = \int_{|y|<1/j} \varphi_j(y)\, g(y)\, dm(y) . \end{aligned}$$

Here $g(y) \to 0$ as $y \to 0$ by 27.3. Hence $\Delta_j \leq \sup_{|y|<1/j} g(y) \to 0$. ∆∆

27.6. <u>THEOREM.</u> Suppose that U is an open set in R^n such that $f|U$ is continuous. Then $f*\varphi_j \to f$ c-uniformly in U.

<u>Proof.</u> Let F be a compact set in U, and let $\varepsilon > 0$. Since f is uniformly continuous on F, there is δ, $0 < \delta < d(F, \partial U)$, such that $|f(x-y) - f(x)| < \varepsilon$ whenever $x \in F$ and $|y| < \delta$. We show that $|f*\varphi_j(x) - f(x)| \leq \varepsilon$ if $x \in F$ and $j > 1/\delta$.

$$\begin{aligned} |f*\varphi_j(x) - f(x)| &= \left|\int f(x-y)\, \varphi_j(y)\, dm(y) - \int f(x)\, \varphi_j(y)\, dm(y)\right| \\ &\leq \int_{|y|<1/j} |f(x-y) - f(x)|\, \varphi_j(y)\, dm(y) \leq \varepsilon . \end{aligned}$$ ∆∆

27.7. THEOREM. Suppose that U is an open set in R^n and that $g : U \to R^1$ is ACL^p. Then there is a sequence of functions $g_j : U \to R^1$ such that

(1) $g_j \in C^1$.

(2) $g_j \to g$ c-uniformly in U.

(3) For each compact set $F \subset U$ and for every $1 \le i \le n$, $\partial_i g_j \to \partial_i g$ in $L^p(F)$.

Proof. We show that for every compact $F \subset U$ there is a sequence of C^1-functions $g_j : R^n \to R^1$ such that $g_j \to g$ uniformly in F and such that $\partial_i g_j \to \partial_i g$ in $L^p(F)$. (This is precisely what we need in the sequel.) The theorem can be obtained from this result by representing U as the union of an expanding sequence of compact sets and by applying the diagonal process.

We cover F by a finite number of balls $B_1, \ldots, B_k$ such that $\bar{B}_i \subset U$. Then $V = B_1 \cup \ldots \cup B_k$ is an open set such that $\bar{V}$ is compact, $F \subset V$, $\bar{V} \subset U$, and $m(\partial V) = 0$. Define $f : R^n \to R^1$ by $f(x) = g(x)$ for $x \in \bar{V}$ and by $f(x) = 0$ for $x \notin \bar{V}$. Then f is integrable, and $f|V = g|V$ is continuous. We can thus form the functions $g_j = f * \varphi_j$. By 27.6, $g_j \to g$ uniformly in F. By 27.2, $g_j \in C^1$ and $\partial_i g_j = f * \partial_i \varphi_j$. On the other hand, $\partial_i g = \partial_i f$ in V, $\partial_i f = 0$ in $\complement\bar{V}$, and $m(\partial V) = 0$. Since $\partial_i g \in L^p$ in $\bar{V}$, this implies $\partial_i f \in L^p$ in R^n. Hence, by 27.5, $\partial_i f * \varphi_j \to \partial_i f$ in L^p. Consequently, it suffices to prove that there is an integer j_0 such that $\partial_i f * \varphi_j(x) = f * \partial_i \varphi_j(x)$ for all $j \ge j_0$, $x \in F$, and $1 \le i \le n$. We show that this is true if $j_0 > \sqrt{n} / d(F, \partial V)$.

Fix $x \in F$, $1 \le i \le n$, and $j \ge j_0$. Let Q be the closed cube $\{x \mid |x_k| \le 1/j\}$ for $1 \le k \le n$. Then $\operatorname{spt} \varphi_j \subset Q$, $x + Q \subset V$, and we obtain by Fubini's theorem

$$\partial_i f * \varphi_j(x) = \int_Q \partial_i f(x-y)\, \varphi_j(y)\, dm(y) = \int_Q \partial_i g(x-y)\, \varphi_j(y)\, dm(y) =$$

$$= \int_{F_i Q} dm_{n-1}(z) \int_{-1/j}^{1/j} \partial_i g(x-z-te_i)\, \varphi_j(z+te_i)\, dt\,.$$

Since g is ACL, the function $t \mapsto g(x-z-te_i)$ is absolutely continuous on $[-1/j\,,\,1/j]$ for almost every $z \in F_i Q$. For such z we may integrate by parts. Observing that $\varphi_j(b) = 0$ for $b \in \partial Q$, this yields

$$\partial_i f * \varphi_j(x) = \int_{F_i Q} dm_{n-1}(z) \int_{-1/j}^{1/j} g(x-z-te_i)\, \partial_i \varphi_j(z+te_i)\, dt$$

$$= \int_Q g(x-y)\, \partial_i \varphi_j(y)\, dm(y) = \int_Q f(x-y)\, \partial_i \varphi_j(y)\, dm(y)$$

$$= f * \partial_i \varphi_j(x)\,. \quad \Delta\!\!\Delta$$

27.8. Remark. By the same method we see that the approximations can be in fact chosen to be C^∞. The converse of 27.7 is also true. It can therefore be used as an alternate definition for ACL^p-functions. A third possibility is to use generalized (=distributional) derivatives. See, for example, Smirnov [1, p. 288].

28. Fuglede's theorem

Roughly speaking, Fuglede's theorem states that an ACL^p-function is absolutely continuous on almost every path. We first give an auxiliary result.

28.1. THEOREM. Suppose that E is a Borel set in R^n and that $f_k : E \to \dot{R}^1$ is a sequence of Borel functions which converge to a Borel function $f : E \to \dot{R}^1$ in $L^p(E)$. Then there is a subsequence $f_{k_1}, f_{k_2}, \ldots$ such that $\int_\gamma |f_{k_j} - f|\, ds \to 0$ for all rectifiable paths γ in E, except for a family Γ such that $M_p(\Gamma) = 0$.

Proof. Choose a subsequence (f_{k_j}) such that

$$\int_E |f_{k_j} - f|^p \, dm < 2^{-pj-j} .$$

Set $g_j = |f_{k_j} - f|$, and let Γ be the family of all rectifiable paths γ such that $|\gamma| \subset E$ and $\int_\gamma g_j \, ds \not\to 0$. We show that $M_p(\Gamma) = 0$.

Let Γ_j be the family of all rectifiable paths γ in E such that $\int_\gamma g_j \, ds > 2^{-j}$. Then $2^j g_j \in F(\Gamma_j)$ if we define $g_j(x) = 0$ for $x \notin E$. Thus

$$M_p(\Gamma_j) \le 2^{pj} \int_E g_j^p \, dm < 2^{-j} .$$

On the other hand, $\Gamma \subset \bigcup_{j=i}^{\infty} \Gamma_j$ for every $i \in N$. Hence

$$M_p(\Gamma) \le \sum_{j=i}^{\infty} M_p(\Gamma_j) < \sum_{j=i}^{\infty} 2^{-j} = 2^{-i+1}$$

for every $i \in N$. Consequently, $M_p(\Gamma) = 0$. $\Delta\!\Delta$

28.2. FUGLEDE'S THEOREM. Suppose that U is an open set in R^n and that $f : U \to R^m$ is ACL^p. Let Γ be the family of all locally rectifiable paths in U which have a closed subpath on which f is not absolutely continuous. Then $M_p(\Gamma) = 0$.

Proof. By considering separately the coordinate functions $f_1, \ldots, f_m$, we may assume that $m = 1$. We express U as the union of an expanding sequence of open sets U_j such that each $\bar{U}_j$ is a compact subset of U. Let Γ_j be the family of all closed paths $\gamma \in \Gamma$ such that $|\gamma| \subset U_j$. Then $\Gamma > \cup \Gamma_j$, whence

$$M_p(\Gamma) \le \sum_{j=1}^{\infty} M_p(\Gamma_j) .$$

It thus suffices to prove that $M_p(\Gamma_j) = 0$ for an arbitrary fixed j.

By 27.7, there is a sequence of C^1-functions $f_k : U \to R^1$ such that $f_k \to f$ uniformly in $\bar{U}_j$ and such that $\partial_i f_k \to \partial_i f$ in $L^p(\bar{U}_j)$, $1 \leq i \leq n$. Passing to a subsequence, we may assume, by 28.1, that

$$\int_\gamma |\partial_i f_k - \partial_i f| \, ds \to 0$$

for all $1 \leq i \leq n$ and for all rectifiable paths γ in U_j except for a family Γ_0 with $M_p(\Gamma_0) = 0$. We show that $\Gamma_j \subset \Gamma_0$, which will prove that $M_p(\Gamma_j) = 0$.

Suppose that $\gamma \in \Gamma_j \setminus \Gamma_0$. Let $\beta = \gamma^0 : [0, c] \to U_j$ be the normal representation 2.5 of γ. We write

$$\beta(t) = \sum_{i=1}^{n} \beta_i(t) e_i .$$

Since $f_k \circ \beta$ is absolutely continuous, we have for every $0 \leq t \leq c$

$$(28.3) \quad f_k(\beta(t)) - f_k(\beta(0)) = \int_0^t (f_k \circ \beta)'(u) \, du = \int_0^t \sum_{i=1}^{n} \partial_i f_k(\beta(u)) \, \beta_i'(u) \, du .$$

Here $|\beta_i'(u)| \leq |\beta'(u)| = 1$ for almost every $u \in [0, c]$, by 1.3. As $k \to \infty$, the left hand side of (28.3) tends to $f(\beta(t)) - f(\beta(0))$. On the other hand,

$$\left| \int_0^t \sum_{i=1}^{n} \partial_i f_k(\beta(u)) \, \beta_i'(u) \, du - \int_0^t \sum_{i=1}^{n} \partial_i f(\beta(u)) \, \beta_i'(u) \, du \right|$$

$$\leq \sum_{i=1}^{n} \int_0^t |\partial_i f_k(\beta(u)) - \partial_i f(\beta(u))| \, |\beta_i'(u)| \, du$$

$$\leq \sum_{i=1}^{n} \int_\gamma |\partial_i f_k - \partial_i f| \, ds \to 0 .$$

Hence (28.3) implies

$$f(\beta(t)) - f(\beta(0)) = \int_0^t \sum_{i=1}^{n} \partial_i f(\beta(u)) \, \beta_i'(u) \, du .$$

As an integral, $f \circ \beta$ is absolutely continuous. In other words, f is absolutely continuous on γ. Since $\gamma \in \Gamma_j \subset \Gamma$, this is a contradiction. ΔΔ

28.4. Remark. Both 28.1 and 28.2 are due to Fuglede [1].

29. The theorem of Rademacher-Stepanov

Let U be an open set in R^n, and let $f : U \to R^m$ be a mapping. In Section 5 we introduced the notation

$$L(x, f) = \limsup_{h \to 0} \frac{|f(x+h) - f(x)|}{|h|} .$$

The most general form (Saks [1, p. 311]) of the theorem of Rademacher-Stepanov states that if $L(x, f) < \infty$ a.e., then f is differentiable a.e. We prove a special case which is adequate for our purposes.

29.1. THE THEOREM OF RADEMACHER-STEPANOV. Suppose that U is an open set in R^n and that $f : U \to R^m$ is a mapping which has the following properties:

(1) f is continuous.

(2) The partial derivatives of f exist a.e.

(3) $L(x, f) < \infty$ a.e.

Then f is differentiable a.e.

Proof. Considering the coordinate functions of f separately, we may assume that $m = 1$. We may also assume that U is bounded. Fix $\delta > 0$, and let A_i denote the set of all points x in U such that $|f(x+h) - f(x)| \leq i\,|h|$ whenever $|h| < 1/i$ and $x+h \in U$. Since f is continuous, each A_i is closed in U. Moreover,

$A_i \subset A_{i+1}$ and $A = \cup A_i = \{x \in U \mid L(x, f) < \infty\}$. Thus $m(U \setminus A) = 0$. Hence there is an integer i_o such that $m(U \setminus A_{i_o}) < \delta/2$. Next consider the functions

$$g_k(x) = \max_{1 \le i \le n} \sup_{0 < |r| < 1/k} \left| \frac{f(x+re_i) - f(x)}{r} - \partial_i f(x) \right| .$$

Then g_k is defined a.e. in U, and $g_k \to 0$ a.e. Since f is continuous, the supremum can be taken over rational r. From this and from 25.2 it follows that g_k is measurable. By Egorov's theorem, there exists a compact set $F \subset U$ such that $m(U \setminus F) < \delta/2$ and such that $g_k | F \to 0$ uniformly. This implies that the functions $\partial_i f | F$ are continuous. Set $E = F \cap A_{i_o}$, and let H be the set of points $x \in E$ which are points of density of E. By the density theorem 23.8, $m(E \setminus H) = 0$. This implies that $m(U \setminus H) < \delta$. We show that f is differentiable at every point of H, which will prove the theorem.

Fix $y \in H$, and set

$$\varepsilon_1(t) = \frac{m(B^n(y, t) \setminus E)}{m(B^n(y, t))},$$

$$\varepsilon_2(t) = \max_{1 \le i \le n} \sup_{\substack{0 < |r| < t \\ x \in F}} \left| \frac{f(x+re_i) - f(x)}{r} - \partial_i f(x) \right| ,$$

$$\varepsilon_3(t) = \max_{1 \le i \le n} \sup_{\substack{|x-y| < t \\ x, y \in F}} |\partial_i f(x) - \partial_i f(y)| .$$

Then $\lim_{t \to 0} \varepsilon_i(t) = 0$ for $i = 1, 2, 3$. Suppose that x is a point in U such that $|x-y| = t < d(y, \partial U)/2$ and $t < 1/2i_o$. Set $z^i = (x_1, \ldots, x_i, y_{i+1}, \ldots, y_n)$, $0 \le i \le n$. In particular, $z^0 = y$ and $z^n = x$. If $r \le t$, then $B^n(z^i, r) \subset B^n(y, 2t) \subset U$ and

$$m(B^n(z^i, r) \setminus E) \le m(B^n(y, 2t) \setminus E) = \varepsilon_1(2t)\, m(B^n(y, 2t))$$
$$< m(B^n(z^i, r))$$

if $r > 2t\,\varepsilon_1(2t)^{1/n}$. We choose now t so small that $\varepsilon_4(t) = 2\,(t + \varepsilon_1(2t)^{1/n}) < 1$ and put $r = \varepsilon_4(t)\,t$. Then $r \leq t$ and $m(B^n(z^i, r) \cap E) > 0$. We can thus find points $u^i \in B^n(z^i, r) \cap E$, $0 \leq i \leq n$. We may put $u^0 = y$. Setting $v^i = u^{i-1} + (x_i - y_i)e_i$, we have $|v^i - z^i| = |u^{i-1} - z^{i-1}| < r$. Moreover,

$$|f(v^i) - f(u^{i-1}) - \partial_i f(u^{i-1})(x_i - y_i)| \leq |x_i - y_i|\,\varepsilon_2(|x_i - y_i|) \leq t\,\varepsilon_2(t) .$$

Since

$$f(x) - f(y) = f(x) - f(u^n) + \sum_{i=1}^{n} (f(u^i) - f(v^i)) + \sum_{i=1}^{n} (f(v^i) - f(u^{i-1})) ,$$

we obtain

$$|f(x) - f(y) - \sum_{i=1}^{n} \partial_i f(y)(x_i - y_i)| \leq |f(x) - f(u^n)| + \sum_{i=1}^{n} |f(u^i) - f(v^i)|$$

$$+ \sum_{i=1}^{n} |f(v^i) - f(u^{i-1}) - \partial_i f(u^{i-1})(x_i - y_i)|$$

$$+ \sum_{i=1}^{n} |\partial_i f(u^{i-1}) - \partial_i f(y)|\,|x_i - y_i|$$

$$\leq i_0 r + 2n i_0 r + nt\varepsilon_2(t) + nt\varepsilon_3(2t)$$

$$= t\,((2n+1)i_0\varepsilon_4(t) + n\varepsilon_2(t) + n\varepsilon_3(2t)) = t\,\varepsilon(t) ,$$

where $\varepsilon(t) \to 0$ as $t \to 0$. Hence f is differentiable at y . ∆∆

29.2. Remark. The above proof is based on Stepanov [1].

30. Hausdorff measure

30.1. We first give the definition of the *α-dimensional Hausdorff outer measure* $\Lambda_\alpha^*(A)$ of a set $A \subset R^n$, $\alpha > 0$. Let $r > 0$. We consider countable coverings $\{A_i \mid i \in N\}$ of A such that each $d(A_i) < r$. Set

$$\Lambda_\alpha^r(A) = \inf \sum_i d(A_i)^\alpha$$

over all such coverings. Then $\Lambda_\alpha^r(A)$ is decreasing in r, and we put

$$\Lambda_\alpha^*(A) = \lim_{r\to 0} \Lambda_\alpha^r(A) = \sup_{r>0} \Lambda_\alpha^r(A) .$$

We shall also use the fact that Λ_α^r, and hence Λ_α^*, can be defined by using only open coverings. This is due to the fact that every set E with $d(E) < r$ can be covered by an open set G such that $d(G) < r$. We state without proof some properties of Λ_α^*.

30.2. <u>THEOREM.</u> Λ_α^* is a metric outer measure. ΔΔ

Hence Λ_α^* defines a class of measurable sets which includes all Borel sets. If A is Λ_α^*-measurable, we write $\Lambda_\alpha^*(A) = \Lambda_\alpha(A)$.

30.3. <u>THEOREM.</u> If $\alpha < \beta$ and if $\Lambda_\alpha^*(A) < \infty$, then $\Lambda_\beta^*(A) = 0$. ΔΔ

30.4. <u>THEOREM.</u> If $A \subset R^1$, then $m_1^*(A) = \Lambda_1^*(A)$. ΔΔ

More generally, $m_n^*(A) = 2^{-n} \Omega_n \Lambda_n^*(A)$ if $A \subset R^n$. The proof of this equality is not quite elementary (Sard [1]). Since we are usually only interested to know whether $\Lambda_\alpha^*(A)$ is zero, positive finite, or infinite, we give the following weaker result:

30.5. <u>THEOREM.</u> If $A \subset R^n$, then $n^{-n/2} \Lambda_n^*(A) \le m_n^*(A) \le \Omega_n \Lambda_n^*(A)$.

<u>Proof.</u> Let $\{G_i \mid i \in N\}$ be an open covering of A such that $d(G_i) < 1$. Then $m^*(A) \le \sum m(G_i) \le \Omega_n \sum d(G_i)^n$. Thus $m^*(A) \le \Omega_n \Lambda_n^1(A) \le \Omega_n \Lambda_n^*(A)$.

Next let G be an open set containing A, and let $r > 0$. We express G as a countable union of closed cubes Q_i such that $d(Q_i) < r$ and $\operatorname{int} Q_i \cap \operatorname{int} Q_j = \emptyset$. Then $\Lambda_n^r(A) \leq \sum d(Q_i)^n = \sum n^{n/2} m(Q_i) = n^{n/2} m(G)$. Thus $\Lambda_n^*(A) \leq n^{n/2} m(G)$, which implies $\Lambda_n^*(A) \leq n^{n/2} m^*(A)$. ∆∆

30.6. COROLLARY. If $A \subset R^n$, then $m_n(A) = 0$ iff $\Lambda_n(A) = 0$, and $m_n^*(A) = \infty$ iff $\Lambda_n^*(A) = \infty$. ∆∆

We next give some criterions for the absolute continuity of a path $\gamma : [a, b] \to R^n$. We need two auxiliary results. Since the diameter of a set does not increase in an orthogonal projection, the following statement follows directly from the definition of Λ_α^*:

30.7. LEMMA. If P is an orthogonal profection of R^n onto a linear submanifold of R^n, then $\Lambda_\alpha^*(PA) \leq \Lambda_\alpha^*(A)$ for all $A \subset R^n$, $\alpha > 0$. ∆∆

30.8. THEOREM. If $A \subset R^n$ is connected, then $d(A) \leq \Lambda_1^*(A)$.

Proof. Let $\varepsilon > 0$, and pick $a, b \in A$ such that $|a-b| > d(A) - \varepsilon$. We may assume that $a = 0$, $b = te_1$, $t > 0$. Let $P : R^n \to R^1$ be the projection $Px = x_1$. Since A is connected, $[0, t] \subset PA$. Using 30.4 and 30.7 we obtain $d(A) - \varepsilon \leq m_1^*(PA) = \Lambda_1^*(PA) \leq \Lambda_1^*(A)$. Since ε was arbitrary, this proves the theorem. ∆∆

30.9. THEOREM. Suppose that $\gamma : [a, b] \to R^n$ is an injective path such that for every $\varepsilon > 0$ there is $\delta > 0$ with the following property: $\Lambda_1(\cup \gamma\Delta_i) < \varepsilon$ whenever $\Delta_1, \ldots, \Delta_k$ are disjoint closed subintervals of (a, b) such that $\sum m(\Delta_i) < \delta$. Then γ is absolutely continuous.

Proof. Let $\varepsilon > 0$, and let δ be the number given by the condition of the theorem. Suppose that $\Delta_i = [a_i , b_i]$, $1 \le i \le k$, are disjoint subintervals of (a , b) such that $\sum m(\Delta_i) < \delta$. By 30.8, $d(\gamma\Delta_i) \le \Lambda_1(\gamma\Delta_i)$. Thus

$$\sum |\gamma(b_i) - \gamma(a_i)| \le \sum d(\gamma\Delta_i) \le \sum \Lambda_1(\gamma\Delta_i) = \Lambda_1(\cup \gamma\Delta_i) < \varepsilon .$$

Since γ is continuous at the end points, this implies that γ is absolutely continuous on $[a , b]$. ΔΔ

30.10. THEOREM. If $\gamma : \Delta \to R^n$ is a path, $\Lambda_1(|\gamma|) \le \ell(\gamma)$.

Proof. We may assume that γ is rectifiable, that Δ is closed and that γ is a normal representation. Fix $r > 0$. Subdivide Δ to intervals $\Delta_1 , \dots , \Delta_k$ such that each $m(\Delta_i) < r$. Then $d(\gamma\Delta_i) \le \ell(\gamma|\Delta_i) = m(\Delta_i) < r$. Furthermore, $\sum d(\gamma\Delta_i) \le \sum m(\Delta_i) = m(\Delta) = \ell(\gamma)$. Consequently, $\Lambda_1^r(|\gamma|) \le \ell(\gamma)$. Letting $r \to 0$ yields $\Lambda_1(|\gamma|) \le \ell(\gamma)$.

ΔΔ

30.11. Remark. If γ is injective, and $|\gamma|$ is hence an arc, then $\Lambda_1(|\gamma|) = \ell(\gamma)$. In fact, for an arbitrary subdivision of Δ to intervals $\Delta_i = [t_{i-1} , t_i]$ we obtain by 30.8

$$\sum |\gamma(t_i) - \gamma(t_{i-1})| \le \sum d(\gamma\Delta_i) \le \sum \Lambda_1(\gamma\Delta_i) = \Lambda_1(|\gamma|) . \quad ΔΔ$$

30.12. THEOREM. Suppose that $\Delta = [a , b]$ and that $\gamma : \Delta \to R^n$ is an injective path satisfying the following conditions:

(1) There is a closed countable set $E \subset \Delta$ such that γ is absolutely on every closed subinterval of $\Delta \setminus E$.

(2) $\int_\Delta |\gamma'(t)| \, dt < \infty$.

Then γ is absolutely continuous on Δ.

Proof. We show that γ satisfies the condition of 30.9. Let $\varepsilon > 0$, and let $\Delta_1, \dots, \Delta_k$ be disjoint closed subintervals of Δ. Then $\Delta_1 \setminus E$ has a countable number of components I_j. Since E is countable, $\Lambda_1(\gamma E) = 0$. From 30.10 and 3.3 we thus obtain

$$\Lambda_1(\gamma\Delta_1) = \sum_j \Lambda_1(\gamma I_j) \le \sum_j \int_{I_j} |\gamma'(t)|\,dt = \int_{\Delta_1} |\gamma'(t)|\,dt .$$

Similarly, $\Lambda_1(\gamma\Delta_i) \le \int_{\Delta_i} |\gamma'(t)|\,dt$ for all $1 \le i \le k$. Summing over all i yields

$$\Lambda_1(\cup\gamma\Delta_i) \le \int_{\cup\Delta_i} |\gamma'(t)|\,dt .$$

Since the function $A \mapsto \int_A |\gamma'(t)|\,dt$ is absolutely continuous, there is $\delta > 0$ such that $\Lambda_1(\cup\gamma\Delta_i) < \varepsilon$ whenever $\sum m(\Delta_i) < \delta$. ΔΔ

30.13. Remark. Theorem 30.12 holds also without the injectivity condition (Saks [1, p. 228]). However, the injectivity of γ is essential in 30.9. For example, the path $\gamma : [-1, 1] \to R^1$, defined by $\gamma(t) = t \sin(1/t)$, satisfies the condition of 30.9 without being absolutely continuous.

30.14. THEOREM. Suppose that $E \subset R^n$, $\Lambda^*_{n-1}(E) < \infty$, and that $p \in N$. Let $P : R^n \to R^{n-1}$ be the projection $Px = x - x_n e_n$, and let A be the set of points $y \in R^{n-1}$ such that $E \cap P^{-1}(y)$ contains at least p points. Then

$$m^*_{n-1}(A) \le \frac{\Omega_{n-1}}{p} \Lambda^*_{n-1}(E) .$$

Proof. For $k \in N$ we let A_k denote the set of points $y \in A$ for which there exist points $x_1, \dots, x_p$ in $E \cap P^{-1}(y)$ such that $|x_i - x_j| \ge 1/k$ for each pair $x_i \ne x_j$. Then $A_k \subset A_{k+1}$, and $A = \cup A_k$. Consequently, $m^*(A) = \lim m^*(A_k)$. It thus suffices to show that

$p\, m^*(A_k) \leq \Omega_{n-1} \Lambda^*_{n-1}(E)$ for an arbitrary fixed k.

Let $\{G_i \mid i \in N\}$ be an open covering of E such that each $d(G_i) < 1/k$. We show that $p\, m^*(A_k) \leq \Omega_{n-1} \sum d(G_i)^{n-1}$, which will prove the theorem, because it implies $p\, m^*(A_k) \leq \Omega_{n-1} \Lambda^{1/k}_{n-1}(E) \leq \Omega_{n-1} \Lambda^*_{n-1}(E)$.

If $y \in A_k$, $P^{-1}(y)$ meets at least p sets G_i. Denoting by g_i the characteristic function of PG_i, we thus have $\sum g_i(y) \geq p$ for all $y \in A_k$. Since each PG_i is open, there is an open set U containing A_k such that $\sum g_i(y) \geq p$ for $y \in U$. Hence we obtain

$$p\, m^*(A_k) \leq p\, m(U) \leq \int_{R^{n-1}} (\sum g_i)\, dm = \sum \int_{R^{n-1}} g_i\, dm = \sum m(PG_i)$$

$$\leq \Omega_{n-1} \sum d(PG_i)^{n-1} \leq \Omega_{n-1} \sum d(G_i)^{n-1}. \quad \Delta$$

30.15. Definition. A set $E \subset R^n$ is said to have a *σ-finite* α-dimensional measure if E is the union of a countable number of sets E_i such that $\Lambda^*_\alpha(E_i) < \infty$.

30.16. THEOREM. Suppose that $E \subset R^n$ has a σ-finite $(n-1)$-dimensional measure and that $P : R^n \to R^{n-1}$ is as in 30.14. Then $E \cap P^{-1}(y)$ is countable for almost every $y \in R^{n-1}$.

Proof. Let A be the set of points $y \in R^{n-1}$ such that $E \cap P^{-1}(y)$ is uncountable. We express E as a countable union of sets E_i such that $\Lambda^*_{n-1}(E_i) < \infty$. Let A_i denote the set of points $y \in R^{n-1}$ such that $E_i \cap P^{-1}(y)$ is uncountable. Since $A = \cup A_i$, it suffices to prove that $m(A_i) = 0$ for all i. Let A_{ip} be the set of points $y \in R^{n-1}$ such that $E_i \cap P^{-1}(y)$ contains at least p points. Since $A_i \subset A_{ip}$, 30.14 implies

$$m^*(A_i) \leq \frac{\Omega_{n-1}}{p} \Lambda^*_{n-1}(E_i).$$

Since this holds for every $p \in N$, $m^*(A_i) = 0$. ΔΔ

30.17. Remark. Theorem 30.14 is due to Gross [1].

CHAPTER 4. THE ANALYTIC PROPERTIES OF QUASICONFORMAL MAPPINGS

In this chapter we first show that the qc mappings have certain analytic properties, for example, that they are ACL^n and a.e. differentiable. Next we show that conversely, certain analytic properties imply that the mapping in question is qc. We thus obtain an analytic characterization for qcty. This is the main result of the chapter. We also give some applications. The chapter consists of sections 31-37.

31. The ACL-property

In this section we show that qc mappings are ACL. An auxiliary result is needed:

31.1. LEMMA. Suppose that F is a compact set in R^1 and that $\varepsilon > 0$. Then there exists $\delta > 0$ with the following property: For every $r \in (0, \delta)$ there exists a finite covering of F with open intervals $\Delta_1, \dots, \Delta_p$ such that

(1) $m(\Delta_i) = 2r$ for $1 \leq i \leq p$.

(2) The center of Δ_i belongs to F.

(3) Each point of F belongs to at most two Δ_i.

(4) $pr < m(F) + \varepsilon$.

Proof. Choose an open set G such that $F \subset G$ and $m(G) < m(F) + \varepsilon$. We show that $\delta = d(F, \complement G)$ has the desired property. Suppose that $0 < r < \delta$. For $x \in F$ set $\Delta(x) = (x-r, x+r)$. Then there exists a finite covering $\{\Delta(x_1), \ldots, \Delta(x_p)\}$ of F such that $x_1 < \ldots < x_p$. This covering satisfies clearly the conditions (1) and (2). If $\Delta(x_i)$ meets $\Delta(x_{i+2})$, we may leave out $\Delta(x_{i+1})$ and obtain a covering which still satisfies (1) and (2). After a finite number of steps, we obtain a covering, which we still denote by $\{\Delta(x_1), \ldots, \Delta(x_p)\}$, which satisfies (1), (2) and (3). We show that it also satisfies (4). Let g_i be the characteristic function of $\Delta(x_i)$. Since $\Delta(x_i) \subset G$, we obtain

$$2pr = \sum m(\Delta(x_i)) = \sum \int_{R^1} g_i \, dm = \int_{R^1} \sum g_i \, dm \leq \int_G 2 \, dm = 2\, m(G)$$
$$< 2\, m(F) + 2\varepsilon . \quad \Delta$$

31.2. <u>THEOREM.</u> Let $f : D \to D'$ be a homeomorphism such that $H(x, f)$ is bounded. Then f is ACL. Here $H(x, f)$ is the linear dilatation, defined in 22.2.

<u>Proof.</u> Let $Q = \{x \mid a_i \leq x_i \leq b_i\}$ be a closed n-interval in $D \setminus \{\infty, f^{-1}(\infty)\}$. Consider, for example, the orthogonal projection $P : R^n \to R^{n-1} = R^{n-1}_n$. For each Borel set $A \subset \operatorname{int} PQ$ we set $E_A = Q \cap P^{-1}A$. Here int means interior with respect to R^{n-1}. Since E_A is a Borel set, fE_A is also a Borel set and hence measurable. Setting $\varphi(A) = m(fE_A)$ we obtain a set function φ in $\operatorname{int} PQ$. By Lebesgue's theorem 23.5, φ has a finite derivative $\varphi'(y)$ for almost every $y \in \operatorname{int} PQ$. Fix such y. We shall prove that f is absolutely continuous on the segment $J = E_y$, which will prove the theorem.

Let F be a compact subset of $J \cap \operatorname{int} Q$. We want to estimate $\Lambda_1(fF)$. Choose H such that $H(x, f) < H$ for all $x \in D$. Let $k > 1/d(F, \partial Q)$, and let F_k be the set of all $x \in F$ such that $0 < r < 1/k$ implies $L(x, f, r) \leq H\, \ell(x, f, r)$ (see 22.1). Then $F_k \subset F_{k+1}$ and $F = \cup F_k$. Since f is continuous, each F_k is compact. Fix k, and choose $\varepsilon > 0$, $t > 0$. Let δ be the number given by Lemma 31.1 for the set F_k. Next choose $r > 0$ such that $r < \min(\delta, 1/k)$ and such that $|f(x) - f(z)| < t$ whenever $x, z \in Q$ and $|x-z| \leq 2r$. Let $\Delta_1, \ldots, \Delta_p$ be the covering of F_k, given by 31.1. Then $\Delta_i \subset J$ for $1 \leq i \leq p$. Let A_i be the open n-ball whose diameter is Δ_i. Then $A_i \subset E_B$ where $B = \bar{B}^{n-1}(y, r)$. Let x_i be the center of Δ_i. Since $x_i \in F_k$, we have $L_i \leq H \ell_i$ where $L_i = L(x_i, f, r)$ and $\ell_i = \ell(x_i, f, r)$. Since $d(fA_i) < t$, we obtain the estimate $\Lambda_1^t(fF_k) \leq \sum d(fA_i) \leq 2 \sum L_i$. By Hölder's inequality this implies

$$\Lambda_1^t(fF_k)^n \leq 2^n p^{n-1} \sum L_i^n \leq 2^n H^n p^{n-1} \sum \ell_i^n$$
$$\leq \frac{2^n H^n (m_1(F_k) + \varepsilon)^{n-1}}{\Omega_n r^{n-1}} \sum m(fA_i) .$$

Since every point in fE_B belongs to at most two fA_i, $\sum m(fA_i) \leq 2\, m(fE_B) = 2\varphi(B)$. Observing that $F_k \subset F$, we obtain

$$\Lambda_1^t(fF_k)^n \leq \frac{2^{n+1} H^n \Omega_{n-1} (m_1(F) + \varepsilon)^{n-1} \varphi(B)}{\Omega_n m(B)} .$$

Letting first $r \to 0$, then $\varepsilon \to 0$, and then $t \to 0$, we obtain $\Lambda_1(fF_k)^n \leq C \varphi'(y) m_1(F)^{n-1}$ where $C = 2^{n+1} H^n \Omega_{n-1}/\Omega_n$. Since fF is the limit of the expanding sequence of compact sets fF_k, $\Lambda_1(fF) = \lim \Lambda_1(fF_k)$. Thus

$$\Lambda_1(fF)^n \leq C \varphi'(y) m_1(F)^{n-1} . \tag{31.3}$$

By 30.9, f is absolutely continuous on J. ΔΔ

By 22.4 we obtain

31.4. <u>COROLLARY.</u> Every qc mapping is ACL. ΔΔ

31.5. <u>Remark.</u> The corollary can be proved somewhat simpler than the theorem. See Remark 34.8.6. The above proof is an n-dimensional version of Gehring [1].

32. Differentiability and the ACL^n-property

In this section we show that qc mappings are a.e. differentiable and ACL^n. We also prove a partial converse which yields an analytic characterization for the outer dilatation $K_O(f)$ of a homeomorphism. The inner dilatation is postponed until Section 34.

32.1. THEOREM. Suppose that $f : D \to D'$ is a homeomorphism such that $H(x, f)$ is bounded. Then f is differentiable a.e.

Proof. Theorems 31.2 and 26.4 imply that the partial derivatives of f exist a.e. By 24.2, f has a.e. a finite volume derivative $\mu_f'(x)$. Consider $x_0 \in D$ such that $x_0 \neq \infty \neq f(x_0)$ and $\mu_f'(x_0) < \infty$. By the theorem 29.1 of Rademacher and Stepanov, it suffices to show that $L(x_0, f) < \infty$.

Since $H(x_0, f) < \infty$, there are positive numbers r_0 and H such that $L(x_0, f, r) \leq H\,\ell(x_0, f, r)$ for $0 < r < r_0$. For all such r we have $\Omega_n L(x_0, f, r)^n \leq H^n m(f\bar{B}^n(x_0, r))$. Consequently,

$$\frac{L(x_0, f, r)^n}{r^n} \leq H^n \frac{m(f\bar{B}^n(x_0, r))}{m(\bar{B}^n(x_0, r))} .$$

Letting $r \to 0$ yields $L(x_0, f)^n \leq H^n \mu_f'(x_0) < \infty$. ∆∆

32.2. COROLLARY. A qc mapping is differentiable a.e. ∆∆

32.3. THEOREM. If $f : D \to D'$ is a homeomorphism and if $1 \leq K < \infty$, the following conditions are equivalent:

(1) $K_O(f) \leq K$.

(2) f is ACL, a.e. differentiable, and $|f'(x)|^n \leq K\,|J(x, f)|$ a.e.

Moreover, each of these conditions implies that f is ACL^n.

<u>Proof.</u> Suppose first that (1) holds. By 22.3, $H(x, f)$ is bounded. By 31.2 and 32.1, f is ACL and a.e. differentiable. The inequality $|f'(x)|^n \leq K|J(x, f)|$ follows from 15.2. Hence (1) implies (2).

Next we show that (2) implies that f is ACL^n. Let E be a compact set in $D \setminus \{\infty, f^{-1}(\infty)\}$. Using 24.2 and 24.4 we obtain

$$\int_E |f'(x)|^n \, dm(x) \leq K \int_E |J(x, f)| \, dm(x) = K \int_E \mu_f'(x) \, dm(x) \leq K \, m(fE) < \infty .$$

Since $|\partial_i f(x)| \leq |f'(x)|$ at every point of differentiability, $\partial_i f \in L^n(E)$. Thus f is ACL^n.

Assume now that (2) holds. To prove (1) we must show that $M(\Gamma) \leq K M(\Gamma')$ for every path family Γ in D. This is done by modifying the proof of the corresponding result 15.1 for diffeomorphisms.

Let Γ_0 denote the family of all locally rectifiable paths $\gamma \in \Gamma$ such that f is absolutely continuous on every closed subpath of γ, and let. $\Gamma_\infty = \{\gamma \in \Gamma \mid \infty \in |f \circ \gamma|\}$. Then $M(\Gamma_\infty) = 0$ by 7.9. Since f is ACL^n, it follows from Fuglede's theorem 28.2 that $M(\Gamma \setminus \Gamma_\infty \setminus \Gamma_0) = 0$. Hence $M(\Gamma_0) = M(\Gamma)$. It thus suffices to prove that $M(\Gamma_0) \leq K M(\Gamma')$.

Let $\rho' \in F(\Gamma')$. Define $\rho : R^n \to \dot{R}^1$ by $\rho(x) = \rho'(x) L(x, f)$ for $x \in D$ and $\rho(x) = 0$ for $x \notin D$. If $\gamma \in \Gamma_0$, 5.3 yields

$$\int_\gamma \rho \, ds \geq \int_{f \circ \gamma} \rho' \, ds \geq 1 .$$

Thus $\rho \in F(\Gamma_0)$, which implies

$$M(\Gamma_0) \leq \int \rho^n \, dm = \int_D \rho'(f(x))^n L(x, f)^n \, dm(x)$$

$$= \int_D \rho'(f(x))^n |f'(x)|^n \, dm(x) \leq K \int_D \rho'(f(x))^n |J(x, f)| \, dm(x) .$$

By 24.4 and 24.5 this implies

$$M(\Gamma_o) \le K \int \varrho'^n \, dm .$$

Since this holds for every $\varrho' \in F(\Gamma')$, $M(\Gamma_o) \le K M(\Gamma')$. ΔΔ

32.4. COROLLARY. A qc mapping is ACL^n. ΔΔ

32.5. Remarks. 1. The proof for 32.1 is an n-dimensional version of Mori [1]. Theorem 32.3 is from Väisälä [1]. The condition (2) in 32.3 can be replaced by the following apparently weaker condition (Gehring [3]): f is ACL, and $L(x, f)^n \le K \mu_f'(x)$ a.e. Indeed, this implies $L(x, f) < \infty$ a.e., and the a.e. differentiability follows from the theorem of Rademacher-Stepanov.

2. Bojarski has proved that every 2-dimensional qc mapping is ACL^p for some $p > 2$ (Lehto-Virtanen [1, p. 226]). It is not known whether an n-dimensional qc mapping, $n \ge 3$, must be ACL^p for some $p > n$.

33. The condition (N)

In this section we prove that every qc mapping satisfies the condition (N), defined in 24.6. The proof is based on the following result of Fubini type:

33.1. THEOREM. Let U be an open set in R^n, let A be a Borel set in U, and let g be the characteristic function of A. Then $m(A) = 0$ iff $\int_\gamma g \, ds = 0$ for almost every closed rectifiable path γ in U.

Proof. Suppose first that $m(A) = 0$. Let Γ be the family of all closed rectifiable paths γ in U such that $\int_\gamma g\, ds > 0$. Define $\varrho : R^n \to \dot{R}^1$ by $\varrho(x) = \infty$ for $x \in A$ and $\varrho(x) = 0$ for $x \notin A$. If $\gamma \in \Gamma$, then $\int_\gamma \varrho\, ds = \infty$. Thus $\varrho \in F(\Gamma)$, whence

$$M(\Gamma) \le \int \varrho^n\, dm = 0 .$$

Conversely, assume that the condition of the theorem is satisfied. It suffices to show that $m(Q \cap A) = 0$ for an arbitrary closed cube Q in U. Let Γ be the family of all line segments J such that J is parallel to the x_n-axis, joins the opposite faces of Q, and $m_1(J \cap A) > 0$. Since $\int_J g\, ds > 0$ for all $J \in \Gamma$, $M(\Gamma) = 0$. (We identify J with its homeomorphic representation.) On the other hand, it follows from Remark 7.3 that $M(\Gamma) = m(E)/h^n$ where h is the length of the edge of Q and E is the union of all segments J in Γ. Thus $m(E) = 0$. By Fubini's theorem, $m(Q \cap A) = 0$. $\Delta\!\!\Delta$

33.2. <u>THEOREM.</u> A qc mapping $f : D \to D'$ satisfies the condition (N).

<u>Proof.</u> Let A be a Borel set in D such that $m(A) = 0$. We must show that $m(fA) = 0$. Let g and g' be the characteristic functions of A and fA, respectively, and let Γ' be the family of all closed rectifiable paths γ in D' such that $\int_\gamma g'\, ds > 0$. By 33.1, it suffices to prove that $M(\Gamma') = 0$. Since f is qc, it suffices to show that $M(\Gamma) = 0$ where $\Gamma = f^{-1}\Gamma'$. Let Γ_0 be the family of all paths $\gamma \in \Gamma$ such that γ is rectifiable and f is absolutely continuous on γ. By 32.4, f is ACL^n. Hence it follows from Fuglede's theorem 28.2 that $M(\Gamma_0) = M(\Gamma)$. It thus suffices to prove that $M(\Gamma_0) = 0$. If $\gamma \in \Gamma_0$, 5.3 implies

$$\int_\gamma g(x)\, L(x, f)\, |dx| \ge \int_{f\circ\gamma} g'\, ds > 0 .$$

Hence also $\int_\gamma g\, ds > 0$ for every $\gamma \in \Gamma_0$. By 33.1, this implies $M(\Gamma_0) = 0$. $\Delta\!\!\Delta$

33.3. THEOREM. If $f : D \to D'$ is qc and if $A \subset D$ is measurable, then fA is measurable, and

$$m(fA) = \int_A |J(x, f)| \, dm(x) .$$

Moreover, $J(x, f) \neq 0$ a.e.

Proof. Since f is differentiable a.e., $|J(x, f)| = \mu_f'(x)$ a.e. The theorem follows from 24.8. ⩟

33.4. Remarks. 1. By Topology, it is impossible that there exist points $a, b \in D$ such that f is differentiable at both points and such that $J(a, f) > 0$, $J(b, f) < 0$. Thus either $J(x, f) > 0$ a.e. or $J(x, f) < 0$ a.e. In the first case, f is sense-preserving, and in the second case sense-reversing.

2. The proof of 33.2 is from Väisälä [1]. Another proof has been given by Gehring [3]. Rešetnjak [1] has proved that, more generally, every ACL^n-homeomorphism satisfies the condition (N).

34. The metric definition and the analytic definition for quasiconformality

In this section we give two new characterizations for qcty. The first one is called the metric definition, because it can be generalized to every metric space.

34.1. THEOREM. (The metric definition for qcty) A homeomorphism $f : D \to D'$ is qc iff $H(x, f)$ is bounded.

Proof. If f is qc, then $H(x, f)$ is bounded by 22.4. Conversely, assume that $H(x, f) \leq H < \infty$ for every $x \in D$. By 31.2, f is

ACL, and by 32.1, f is a.e. differentiable. If f is differentiable at x, then either $f'(x) = 0$ or $0 < H(f'(x)) = H(x, f) \le H$. In both cases $|f'(x)|^n \le H^{n-1} |J(x, f)|$. From 32.3 it thus follows that $K_O(f) \le H^{n-1}$. Since $K_I(f^{-1}) = K_O(f)$, it follows from 22.3 that $H(y, f^{-1})$ is bounded for $y \in D'$. Repeating the above argument we conclude that $K_I(f) = K_O(f^{-1}) < \infty$. Hence f is qc. ∆∆

34.2. <u>Remark.</u> From the above proof we obtain a somewhat stronger result: For each $K \ge 1$ and $n \ge 2$ there is H such that $H(x, f) \le H$ for every n-dimensional K-qc mapping f. Similarly, for each $H \ge 1$ and $n \ge 2$ there is K such that $K(f) \le K$ for every n-dimensional homeomorphism f which satisfies the condition $H(x, f) \le H$. The best bound for K is H^{n-1}, as easily follows from the analytic definition 34.4 and from (14.3). The best bound for H is more complicated. However, the analytic definition and (14.3) also imply that $H(x, f) \le K(f)^{2/n}$ a.e. and that this inequality is best possible.

34.3. <u>THEOREM.</u> A homeomorphism $f : D \to D'$ is qc iff one of the dilatations $K_I(f)$, $K_O(f)$ is finite.

<u>Proof.</u> This follows directly from 22.3 and 34.1. ∆∆

34.4. <u>THEOREM.</u> (The analytic definition for the dilatations) Suppose that $f : D \to D'$ is a homeomorphism. If the conditions

(1) f is ACL,

(2) f is differentiable a.e.,

(3) $J(x, f) \ne 0$ a.e.,

are satisfied, then

$$K_I(f) = \operatorname*{ess\,sup}_{x \in D} H_I(f'(x)), \quad K_O(f) = \operatorname*{ess\,sup}_{x \in D} H_O(f'(x)).$$

If one of the conditions (1), (2), (3), is not satisfied, then

$K_I(f) = K_O(f) = \infty$.

Proof. The last assertion follows from 34.3, 31.4, 32.2 and 33.3. Assume next that f satisfies the conditions (1), (2), (3). Then the formula for $K_O(f)$ follows from 32.3. If $K_I(f) < \infty$, 34.3 implies that f is qc. Thus f^{-1} also satisfies the conditions (1), (2), (3), and

$$K_I(f) = K_O(f^{-1}) = \operatorname*{ess\,sup}_{y \in D'} H_O(f^{-1\prime}(y)) .$$

On the other hand, if $J(x, f) \neq 0$ and if $y = f(x)$, then $f^{-1\prime}(y) = f'(x)^{-1}$. Hence $H_O(f^{-1\prime}(y)) = H_I(f'(x))$. Since f and f^{-1} satisfy the condition (N) by 33.2,

$$\operatorname*{ess\,sup}_{y \in D'} H_O(f^{-1\prime}(y)) = \operatorname*{ess\,sup}_{x \in D} H_I(f'(x)) .$$

This proves the formula for $K_I(f)$ in the case $K_I(f) < \infty$. Finally, assume that $K_I(f) = \infty$. By 34.3, also $K_O(f) = \infty$. From (14.3) we obtain

$$\operatorname*{ess\,sup}_{x \in D} H_I(f'(x)) \geq \operatorname*{ess\,sup}_{x \in D} H_O(f'(x))^{1/(n-1)} = K_O(f)^{1/(n-1)} = \infty . \quad \Delta\!\Delta$$

34.5. COROLLARY. If $f : D \to D'$ is a homeomorphism, then

$$1 \leq K_O(f) \leq K_I(f)^{n-1}, \quad 1 \leq K_I(f) \leq K_O(f)^{n-1}.$$

In particular, $K_I(f) = K_O(f)$ for $n = 2$. $\Delta\!\Delta$

34.6. THEOREM. (The analytic definition for K-qcty) A homeomorphism $f : D \to D'$ is K-qc iff the following conditions are satisfied:

(1) f is ACL.

(2) f is differentiable a.e.

(3) For almost every $x \in D$,

$$|f'(x)|^n / K \leq |J(x, f)| \leq K\, \ell(f'(x))^n .$$

Proof. From 34.4 it immediately follows that a K-qc mapping satisfies the above conditions (1), (2), (3). Conversely, assume that these conditions are satisfied. From 32.3 it follows that $K_O(f) \leq K$. By 34.3, f is qc. Hence $J(x, f) \neq 0$ a.e., and the inequality $K_I(f) \leq K$ follows from 34.4. ∆∆

34.7. THEOREM. Suppose that $f : D \to D'$ is a homeomorphism such that every point in D has a neighborhood U such that $K_I(f|U) \leq a$ and $K_O(f|U) \leq b$. Then $K_I(f) \leq a$ and $K_O(f) \leq b$.

Proof. By 26.3, f is ACL. The theorem follows from the analytic definition 34.4. ∆∆

34.8. Remarks. 1. As in 32.5, we can replace the conditions (2) and (3) of 34.6 by the single condition $L(x, f)^n/K \leq \mu_f'(x) \leq K\,\ell(x, f)^n$ a.e., where $\ell(x, f) = \liminf_{h \to 0} |f(x+h) - f(x)|/|h|$.

2. One can also replace the conditions (1) and (2) in 34.6 by the requirement that f be ACL^n. By $f'(x)$ we then mean the linear mapping defined by $f'(x)e_i = \partial_i f(x)$. This is possible, because every ACL^n-homeomorphism is differentiable a.e. (Väisälä [3]).

3. If $n = 2$, every ACL-homeomorphism is differentiable a.e. (Lehto-Virtanen [1, p. 134]). Hence, the condition (2) of 34.6 can be completely left out. Moreover, (3) reduces to the single inequality $|f'(x)|^2 \leq K\,|J(x, f)|$. It is not known to the author whether (2) can be omitted in dimensions $n \geq 3$.

4. It is natural to ask whether 34.6 is true without the condition (1). We show by an example that this is not the case. Let $g : [0, 1] \to R^1$ be a continuous increasing function such that $g(0) = 0$, $g(1) = 1$, and $g'(t) = 0$ a.e. See, for example, Munroe [1, p. 193]. Let D be the open unit cube $0 < x_i < 1$, and set $f(x) = (x_1 + g(x_1), x_2, \ldots, x_n)$. Then f maps D onto the n-interval

$0 < x_1 < 2$, $0 < x_i < 1$, $2 \leq i \leq n$. Moreover, f is differentiable a.e., and $|f'(x)| = \ell(f'(x)) = J(x, f) = 1$ a.e. However, f is not qc, because it is not ACL. Another example is given by the mapping $f(x) = (x_1 + g(x_2), x_2, \ldots, x_n)$.

5. We proved in 32.3 that $K_O(f) \leq K$ is equivalent to the conditions (1), (2) and the first part of (3) of 34.6. It is therefore natural to ask whether (1), (2) and the second part of (3) imply $K_I(f) \leq K$. The following example, pointed to the author by O. Martio, shows that this is not the case. Let E be a Cantor set (totally disconnected perfect set) in $[0, 1]$ such that $m_1(E) > 0$. Define $h(t) = 0$ for $t \in E$ and $h(t) = 1$ for $t \in [0, 1] \setminus E$. Setting

$$g(t) = \int_0^t h(u)\, du$$

we obtain an absolutely continuous strictly increasing function $g : [0, 1] \to R^1$. Let D again be the unit cube of R^n, and set $f(x) = (g(x_1), x_2, \ldots, x_n)$. Then f is an ACL-homeomorphism of D. Since $g'(t) = h(t)$ a.e., either $\ell(f'(x)) = J(x, f) = 0$ or $\ell(f'(x)) = J(x, f) = 1$ a.e. in D. Thus f satisfies (1), (2), and the second part of (3) with $K = 1$. However, f is not qc, since $J(x, f) = 0$ in a set of positive measure.

6. We describe an alternate way of proving the analytic properties of a qc mapping. This proof makes use of neither the theorem of Rademacher-Stepanov nor the linear dilatation. Suppose that $f : D \to D'$ is K-qc. First one can show by a method of Pfluger [1] that f is ACL. The n-dimensional version is given in Väisälä [3]. In fact, we obtain an inequality of the type (31.3). By a method of Agard [1] one can then show that f is not only ACL but in fact ACL^n. This implies that f is differentiable a.e. (Remark 2 above).

The proof arrangement given in these notes is from Väisälä [1]. A slightly different proof has been given by Gehring [3].

7. Pfluger's ACL proof makes only use of path families associ-

ated to right cylinders, such as in Example 7.2. This is also the case with Theorem 15.2. This leads to the following result: A homeomorphism $f: D \to D'$ is K-qc iff $M(\Gamma)/K \le M(\Gamma') \le K\,M(\Gamma)$ for every path family Γ associated to a right cylinder in D.

35. Exceptional sets and the reflection principle

It is often convenient to construct a qc mapping piecewise. The question arises whether such a mapping is really qc. We prove a general result which applies to several important cases. As an application we prove the reflection principle for qc mappings.

35.1. THEOREM. Suppose that $f: D \to D'$ is a homeomorphism and that $E \subset D$ is a set such that E is closed in D and such that E has a σ-finite $(n-1)$-dimensional measure. Suppose also that every point in $D \setminus E$ has a neighborhood U such that $K_I(f|U) \le a$ and $K_O(f|U) \le b$. Then $K_I(f) \le a$ and $K_O(f) \le b$.

Proof. Since E has a σ-finite $(n-1)$-dimensional measure, $m_n(E) = 0$. From the analytic definition 34.4 it thus follows that at almost every point $x \in D$, f is differentiable, $J(x, f) \ne 0$, and $H_I(f'(x)) \le a$, $H_O(f'(x)) \le b$. Hence it suffices to prove that f is ACL.

Let Q be a closed n-interval in $D \setminus \{\infty, f^{-1}(\infty)\}$. Let $P: R^n \to R^{n-1}$ be the orthogonal projection, and set $J_y = Q \cap P^{-1}(y)$ for $y \in PQ$. By symmetry, it suffices to prove that f is absolutely continuous on J_y for almost every $y \in PQ$.

Let A denote the set of all $y \in PQ$ such that $J_y \cap E$ is uncountable. By 30.16, $m_{n-1}(A) = 0$. Next let B be the set of all $y \in PQ$ such that $\int_{J_y} |\partial_n f|\, dm_1$ is either infinite or not defined.

We show that $m_{n-1}(B) = 0$. By Fubini's theorem it suffices to show that $|\partial_n f|$ is integrable over Q. Since $|\partial_n f(x)|^n \le |f'(x)|^n \le b\,|J(x, f)| = b\,\mu_f'(x)$ for almost every $x \in D$, Theorem 24.2 implies

$$\int_Q |\partial_n f|^n \, dm \le b\, m(fQ) < \infty .$$

Thus $|\partial_n f|^n$ is integrable over Q. Since $m(Q) < \infty$, also $|\partial_n f|$ is integrable over Q. Hence $m_{n-1}(B) = 0$.

We choose a sequence of closed n-intervals $S_1, S_2, \ldots$ such that $S_i \subset D \setminus E$ and $D \setminus E = \bigcup \operatorname{int} S_i$. Let C_i be the set of all $y \in PQ$ such that f is absolutely continuous on $S_i \cap P^{-1}(y)$. Since f is ACL in $D \setminus E$, $m_{n-1}(C_i) = 0$ for $i \in N$. Setting $F = A \cup B \cup C_1 \cup C_2 \cup \ldots$ we thus have $m_{n-1}(F) = 0$. We show that f is absolutely continuous on J_y for every $y \in PQ \setminus F$. Since $f|J_y$ is an injective path (we identify J_y with an interval of R^1), we may use 30.12. Since $y \notin A$, $J_y \cap E$ is countable. Since $y \notin B$, $|\partial_n f|$ is integrable over J_y. Finally, let J be a closed subinterval of $J_y \setminus E$. Then J can be covered with a finite number of n-intervals S_i. Since $y \notin C_i$, f is absolutely continuous on every $J \cap S_i$, and hence on J. From 30.12 it follows that f is absolutely continuous on J_y. ∆

We next give the reflection principle. Suppose that $f : D \to D'$ is a qc mapping and that S is a sphere or a hyperplane. Suppose also that $\emptyset \ne E \subset S \cap \partial D$ and that E is relatively open in both S and ∂D. Let $g : \bar{R}^n \to \bar{R}^n$ be the reflection in S. We further assume that $D \cap gD = \emptyset$. It is easy to verify that $D \cup E \cup gD$ is a domain. Let S', g', and E' have the corresponding meaning with respect to D'. Suppose also that E' is the cluster set $C(f, E)$. From 17.17 it follows that f has a unique extension to a homeomorphism $f^* : D \cup E \to D' \cup E'$. We extend f^* to a homeomorphism $f_1 : D \cup E \cup gD \to D' \cup E' \cup g'D'$ by setting $f_1(x) = g'(f(g(x)))$ for $x \in gD$. We say that <u>f is extended to f_1 by reflection.</u>

35.2. <u>THEOREM.</u> (The reflection principle) If a qc mapping f is extended to f_1 by reflection, then f_1 is qc and has the same dilatations as f.

<u>Proof.</u> Since g and g' are conformal, $f_1|gD$ has the same dilatations as f. Since S has a σ-finite $(n-1)$-dimensional measure, 35.1 implies that $K_I(f_1) = K_I(f)$ and $K_O(f_1) = K_O(f)$. ∆∆

35.3. <u>THEOREM.</u> Let $n \geq 3$, let D be the half space $x_n > 0$, and let $f : D \to D$ be K-qc. Then the induced boundary mapping $f_o : \partial D \to \partial D$ is an $(n-1)$-dimensional K_1-qc mapping, where K_1 depends only on n and K.

<u>Proof.</u> We extend f by reflection to a K-qc mapping $f_1 : \bar{R}^n \to \bar{R}^n$. Suppose that $x_o \in \partial D$. By 34.2, $H(x_o, f_o) \leq H(x_o, f_1) \leq H$ where H depends only on n and K. Hence, by 34.2, $K(f_o) \leq H^{n-2}$. ∆∆

We next prove a local extension theorem for qc mappings, which will be needed in Section 40.

35.4. <u>Definition.</u> (cf. 17.5.(5)) A domain D is said to be <u>locally quasiconformally bi-collared</u> at a boundary point b, if there is a neighborhood U of b and a qc mapping $g : U \to B^n$ such that $g(U \cap D)$ is the upper half ball B^n_+.

(From this it follows that $g(U \cap \partial D) = B^{n-1}$.)

35.5. <u>THEOREM.</u> Suppose that D and D' are domains which are locally qclly bi-collared at points $b \in \partial D$ and $b' \in \partial D'$, respectively. Suppose also that $f : D \to D'$ is a qc mapping such that $b' \in C(f, b)$. Then there are neighborhoods U of b and U' of b' and a qc mapping $h : U \to U'$ such that $h(x) = f(x)$ for $x \in U \cap D$.

Proof. By 17.10 and 17.15, $b' = \lim_{x \to b} f(x)$. Choose a neighborhood V' of b' and a qc mapping $g' : V' \to B^n$ such that $g'(V' \cap D') = B^n_+$. Next choose a neighborhood U of b and a qc mapping $g : U \to B^n$ such that $g(U \cap D) = B^n_+$ and $f(\bar{U} \cap D) \subset V'$. Since D and D' are locally qclly collared at every point of $U \cap \partial D$ and $V' \cap \partial D'$, respectively, it follows from 17.17 that f can be extended to a continuous injective mapping $f^* : D \cup (U \cap \partial D) \to D' \cup (V' \cap \partial D')$. Then $f_1 = g' \circ f \circ g^{-1}$ is a qc mapping of B^n_+ onto a domain $G \subset B^n_+$, and $f_1^* = g' \circ f^* \circ g^{-1}$ is a continuous injective extension of f_1 to $B^n_+ \cup B^{n-1}$. By Topology, $f_1^* B^{n-1}$ is an open set in B^{n-1}. We can therefore extend f_1 by reflection to a qc mapping $f_2 : B^n \to G_1$. Then $h = g'^{-1} \circ f_2 \circ g : U \to g'^{-1} G_1$ is the desired mapping. ∆∆

35.6. Remarks. 1. Theorem 35.1 is an n-dimensional version of Strebel [1].

2. Theorem 35.3 is due to Gehring [3]. Gehring has recently proved that $K_I(f_o) \le K_I(f)$ and $K_O(f_o) \le K_O(f)$, and that these bounds are sharp. The case $n = 3$ is treated in Gehring-Väisälä [2, p. 29]. Ahlfors [1] has proved the following converse of 35.3: If $n = 2$, then every qc mapping $f_o : \bar{R}^n \to \bar{R}^n$ can be extended to a qc mapping $f : \bar{R}^{n+1} \to \bar{R}^{n+1}$. It is not known whether this is true for $n \ge 3$. It has an important analogue in the case $n = 1$, proved by Beurling-Ahlfors [1]. One can also consider the qcty of mappings on boundary surfaces (Gehring-Väisälä [2, p. 21]).

36. The ring definition for quasiconformality

36.1. THEOREM. If $f : D \to D'$ is a homeomorphism, then

$$K_I(f) = \sup \frac{M(\Gamma'_A)}{M(\Gamma_A)} , \qquad K_O(f) = \sup \frac{M(\Gamma_A)}{M(\Gamma'_A)} ,$$

where the suprema are taken over all rings A such that $\bar{A} \subset D$ and

$M(\Gamma_A) > 0$.

Proof. Applying the second inequality to the inverse mapping f^{-1} gives the first one. Therefore it suffices to prove the formula for $K_O(f)$. Set $K = \sup M(\Gamma_A) / M(\Gamma'_A)$. Since $K_O(f) \geq K$ trivially, it remains to show that $K_O(f) \leq K$.

We may assume that $K < \infty$. By 22.3, $H(x, f)$ is bounded. By the metric definition 34.1, f is qc. Let x_o be a point in D such that f is differentiable at x_o and $J(x_o, f) \neq 0$. By the analytic definition 34.4, it suffices to show that $|f'(x_o)|^n \leq K |J(x_o, f)|$.

Performing a preliminary similarity transformation we may assume that $x_o = 0 = f(x_o)$ and that $f'(0)$ is given by $f'(0)e_i = a_i e_i$ where $a_1 \geq a_2 \geq \ldots \geq a_n > 0$. We must show that $a_1^{n-1} \leq K a_2 \ldots a_n$. Choose $h > 0$ and $\delta > 0$ such that the closure of the interval $Y = \{x \mid |x_1| < \delta h/a_1 , |x_i| < h + \delta h/a_i$ for $2 \leq i \leq n\}$ is contained in D . Let Q be the closed (n-1)-cube $\{x \mid x_1 = 0 , |x_i| \leq h$ for $2 \leq i \leq n\}$. Then $A = Y \setminus Q$ is a ring with boundary components Q and ∂Y . Let Γ_1 be the family of all $\gamma \in \Gamma_A$ such that $|\gamma|$ is contained in the interval $\{x \mid 0 \leq x_1 \leq \delta h/a_1 , |x_i| \leq h$ for $2 \leq i \leq n\}$, and let Γ_2 be the symmetric image of Γ_1 in $x_1 = 0$. Then Γ_1 and Γ_2 are separate, and $\Gamma_i \subset \Gamma_A$. By 6.7 and 7.2 we obtain

$$M(\Gamma_A) \geq M(\Gamma_1) + M(\Gamma_2) = 2^n a_1^{n-1} / \delta^{n-1} .$$

Next consider the ring $A' = fA$. Fix $0 < \delta < 1$, and choose $0 < \varepsilon < \delta/2$. Next choose h such that $|f(x) - f'(0)x| < \varepsilon h$ for $x \in \bar{Y}$. Then the distance between the boundary components of A' is at least $(\delta - 2\varepsilon)h$. On the other hand, A' is contained in the interval $\{x \mid |x_1| < (\delta + \varepsilon)h , |x_i| < (a_i + \delta + \varepsilon)h$ for $2 \leq i \leq n\}$. Using the inequality $M(\Gamma_A) \leq K M(\Gamma_{A'})$ and 7.1 we obtain

$$a_1^{n-1}(\delta - 2\varepsilon)^n \leq K\,\delta^{n-1}(\delta + \varepsilon)(a_2 + \delta + \varepsilon)\ldots(a_n + \delta + \varepsilon).$$

Letting first $\varepsilon \to 0$ and then $\delta \to 0$ yields $a_1^{n-1} \leq K\,a_2 \ldots a_n$. ⩟

36.2. <u>COROLLARY.</u> A homeomorphism $f : D \to D'$ is K-qc iff the double inequality

$$M(\Gamma_A)/K \leq M(\Gamma_A') \leq K\,M(\Gamma_A)$$

holds for every ring A such that $\bar{A} \subset D$. ⩟

36.3. <u>Remark.</u> Theorem 36.1 is an n-dimensional version of Gehring-Väisälä [1].

37. Quasiconformality of a limit mapping

Suppose that $f_j : D \to D_j$ is a sequence of K-qc mappings converging c-uniformly to $f : D \to \bar{R}^n$. In Section 21 we proved that f is either constant or a homeomorphism onto a domain D'. In this section we complete the discussion by showing that in the latter case f is actually K-qc. The proof is based on the ring definition 36.1. We need some knowledge on the fact that $M(\Gamma_A)$ depends continuously on the ring A. A complete result is given in Gehring [3, p. 367]; we prove only what we need.

37.1. <u>THEOREM.</u> Suppose that C_0 and C_1 are disjoint nondegenerate compact connected sets in $\bar{R}^n$. Then

$$M(\Gamma) = \lim_{r\to 0} M(\Gamma(r)),$$

where $\Gamma = \Delta(C_0, C_1, R^n)$ and $\Gamma(r) = \Delta(C_0 + r\bar{B}^n, C_1 + r\bar{B}^n, R^n)$.

<u>Proof.</u> Set $\Gamma_0(r) = \Delta(C_0 + r\bar{B}^n, C_1, R^n)$. We first prove that if

$\varrho \in F(\Gamma)$ and if $\varrho \in L^n$, then for every $q > 1$ there is $r > 0$ such that $q\varrho \in F(\Gamma_0(r))$. Suppose that this assertion is false. Choose $r_0 > 0$ such that $2r_0 < d(C_0)$ and $2r_0 < d(C_0, C_1)$. For $0 < r < r_0$ choose a rectifiable $\gamma \in \Gamma_0(r)$, $\gamma : [a, b] \to R^n$, such that $\gamma(a) \in C_0 + r\bar{B}^n$, $\gamma(b) \in C_1$, and

$$\int_\gamma \varrho \, ds < 1/q .$$

Put $\Gamma_1 = \Delta(C_0, |\gamma|, R^n)$. Then there is $x_0 \in C_0$ such that $d(x, |\gamma|) \leq r$. If $r < t < r_0$, the sphere $S^{n-1}(x, t)$ meets both C_0 and $|\gamma|$. Hence 10.12 implies that $M(\Gamma_1) \geq c_n \log(r_0/r)$. Let $\gamma_1 : [a_1, b_1] \to R^n$ be an arbitrary rectifiable path in Γ_1 such that $\gamma_1(a_1) \in C_0$ and $\gamma_1(b_1) \in |\gamma|$. Then $\gamma_1(b_1) = \gamma(t_0)$ for some $t_0 \in [a, b]$. Define $\alpha : [a_1, b_1+b-t_0] \to R^n$ by $\alpha(t) = \gamma_1(t)$ for $a_1 \leq t \leq b_1$ and $\alpha(t) = \gamma(t+t_0-b_1)$ for $b_1 \leq t \leq b+b_1-t_0$. Then $\alpha \in \Gamma$, whence

$$1 \leq \int_\alpha \varrho \, ds \leq \int_{\gamma_1} \varrho \, ds + \int_\gamma \varrho \, ds < \int_{\gamma_1} \varrho \, ds + 1/q .$$

Thus $q\varrho/(q-1) \in F(\Gamma_1)$, which implies

$$c_n \log \frac{r_0}{r} \leq M(\Gamma_1) \leq \left(\frac{q}{q-1}\right)^n \int \varrho^n \, dm .$$

Letting $r \to 0$ yields a contradiction.

If $0 < t < r$, then $\Gamma \subset \Gamma(t) \subset \Gamma(r)$, whence $M(\Gamma) \leq M(\Gamma(t)) \leq M(\Gamma(r))$. Hence the limit $p = \lim_{r \to 0} M(\Gamma(r))$ exists and $p \geq M(\Gamma)$. Suppose that $p > M(\Gamma)$. Choose $\varrho \in F(\Gamma)$ and $q > 1$ such that

$$q^{2n} \int \varrho^n \, dm < p .$$

By the first part of the proof, there is $r_0 > 0$ such that $q\varrho \in F(\Gamma_0(r_0))$. Replacing C_0 by C_1 and C_1 by $C_0 + r\bar{B}^n$, we can similarly find r_1 such that $0 < r_1 < r_0$ and $q^2\varrho \in F(\Gamma(r_0, r_1))$ where $\Gamma(r_0, r_1) = \Delta(C_0 + r_0\bar{B}^n, C_1 + r_1\bar{B}^n, R^n)$. Since $\Gamma(r_1) \subset \Gamma(r_0, r_1)$, we obtain

$$p \leq M(\Gamma(r_1)) \leq M(\Gamma(r_0, r_1)) \leq q^{2n} \int \varrho^n \, dm < p ,$$

which is a contradiction. $\Delta\!\Delta$

37.2. <u>THEOREM.</u> Suppose that $f_j : D \to D_j$ is a sequence of qc mappings which converge pointwise to a homeomorphism $f : D \to D'$. Then

$$K_I(f) \leq \liminf_{j \to \infty} K_I(f_j)\,, \qquad K_O(f) \leq \liminf_{j \to \infty} K_O(f_j)\,.$$

<u>Proof.</u> We first prove the inequality for K_O. We may assume that $\liminf K_O(f_j) = K < \infty$. We may also assume that $\infty \notin D$ and $\infty \notin D'$, because we may consider restrictions of f_j to $D \setminus \{\infty\,, f^{-1}(\infty)\}$ and then use 17.3. Let A be a ring such that $\bar{A} \subset D$ and $M(\Gamma_A) > 0$. Set $A_j = f_j A$ and $A' = fA$. By the ring definition 36.1, it sufficies to show that $M(\Gamma_A) \leq K\, M(\Gamma_{A'})$.

Choose $\varepsilon > 0$. Passing to a subsequence we may assume that $K_O(f_j) < K + \varepsilon$ for all j. By 21.1 and 34.5, the convergence is c-uniform. Let B_0, B_1 be the boundary components of A, and let $A' = R(C_0', C_1')$ with $fB_i \subset C_i'$. Choose $r > 0$. Since $f_j \to f$ uniformly in $\bar{A}$, there is j such that $f_j B_0 \subset fB_0 + r\bar{B}^n$ and $f_j B_1 \subset fB_1 + r\bar{B}^n$. Since $\Gamma_{A_j} \subset \Gamma(r) = \Delta(C_0' + r\bar{B}^n\,, C_1' + r\bar{B}^n\,, R^n)$, we obtain $M(\Gamma_A) \leq K_O(f_j)\, M(\Gamma_{A_j}) < (K+\varepsilon)\, M(\Gamma(r))$. As $r \to 0$, 37.1 and 11.3 yield $M(\Gamma_A) \leq (K+\varepsilon)\, M(\Gamma_{A'})$. Since ε was arbitrary, this proves the inequality for K_O.

We next turn to K_I. We may assume that $\liminf K_I(f_j) = K < \infty$. Choose a domain G such that $\bar{G} \subset D'$. By 34.7, it suffices to prove that $K_O(f^{-1}|G) \leq K$. By 21.10, $f_j^{-1}|G$ are defined for large j, and $f_j^{-1} \to f^{-1}$ uniformly in G. Hence, the first part of the proof implies $K_O(f^{-1}|G) \leq \liminf K_O(f_j^{-1}|G) \leq K$. $\Delta\!\Delta$

37.3. <u>COROLLARY.</u> Suppose that $f_j : D \to D_j$ is a sequence of K-qc mappings which converge pointwise to a homeomorphism $f : D \to D'$. Then f is also K-qc. $\Delta\!\Delta$

37.4. COROLLARY. The word "homeomorphism" can be replaced by "K-qc mapping" in the results 21.1, 21.3, 21.5, 21.7, 21.9, and 21.11. ΔΔ

37.5. Remark. Theorem 37.2 was first proved by Ahlfors [1] in the case $n = 2$, and by Gehring [3] and Väisälä [1] in higher dimensions.

A purely analytic proof, which applies also to the larger class of quasiregular mappings, has been given by Rešetnjak [3].

CHAPTER 5. MAPPING PROBLEMS

In this chapter we shall consider two problems: Given two domains D and D' in $\overline{R}^n$, does there exist a qc mapping $f : D \to D'$? Next, if the answer is affirmative, how small can the dilatations of such a mapping be? The chapter consists of sections 38-41.

38. The coefficients of quasiconformality

38.1. Definition. Let D and D' be domains in $\overline{R}^n$, homeomorphic to each other. The *inner coefficient of quasiconformality* of D with respect to D' is the number $K_I(D, D') = \inf K_I(f)$ over all homeomorphisms $f : D \to D'$. Similarly, the *outer coefficient* is $K_O(D, D') = \inf K_O(f)$. If $D' = B^n$, we abbreviate $K_I(D, B^n) = K_I(D)$ and $K_O(D, B^n) = K_O(D)$.

Obviously, $1 \le K_I(D, D'), K_O(D, D') \le \infty$, and

$$K_I(D, D') = K_O(D', D). \tag{38.2}$$

The coefficients are finite iff the domains are qclly equivalent. From 34.5 we obtain

$$K_I(D, D') \le K_O(D, D')^{n-1}, \quad K_O(D, D') \le K_I(D, D')^{n-1}. \tag{38.3}$$

A mapping $f : D \to D'$ is *extremal* for K_I or K_O if $K_I(f) = K_I(D, D')$ or $K_O(f) = K_O(D, D')$, respectively. The extremal mappings do not always exist. If they exist, they are not necessarily unique. However, we can prove the existence in the following special cases:

38.4. THEOREM. The extremal mappings for K_I and K_O exist in the following cases:

(1) D or D' is a ball.

(2) ∂D has exactly k components where $2 \le k < \infty$.

Proof. (1): By (38.2), we may assume that $D' = B^n$. We prove that there is an extremal mapping for K_I ; the proof for K_O is similar.

We may assume that $K_I(D) < \infty$. Fix $x_0 \in D$ and choose a sequence of qc mappings $f_j : D \to B^n$. such that $K_I(f_j) \to K_I(D)$. Since B^n can be mapped conformally onto itself such that a given point is mapped into the origin, we may choose f_j so that $f_j(x_0) = 0$. By 19.3 and 20.5, $\{f_j \mid j \in N\}$ is a normal family. Passing to a subsequence, we may therefore assume that $f_j \to f$ c-uniformly in D . By 21.11, f is either a homeomorphism onto B^n or a constant $c \in \partial B^n$. The latter case is impossible, because $f(x_0) = 0$. From 37.2 it follows that $K_I(f) \leq K_I(D)$. Hence $K_I(f) = K_I(D)$.

(2): We again prove the existence of a mapping $f : D \to D'$ such that $K_I(f) = K_I(D , D')$. We may assume that $K_I(D , D') < \infty$. As in the case (1), we can find a sequence of qc mappings $f_j : D \to D'$ such that $K_I(f_j) \to K_I(D , D')$ and such that $f_j \to f$ c-uniformly. If ∂D contains at least three points, it follows from 21.11 that f is a homeomorphism onto D'. By 37.2, $K_I(f) \leq K_I(D , D')$. If ∂D consists of exactly two points, there is a Möbius transformation $g : D \to D'$, and hence $K_I(D , D') = 1 = K_I(g)$. ΔΔ

38.5. Remark. The proof of (2) shows that the extremal mappings exist if it is not possible that a sequence of K-qc mappings $f_j : D \to D'$ converge c-uniformly to a constant. In addition to the cases mentioned in 21.11, this happens if the domains have certain homotopy properties, for example, if D and D' are tori in R^3. The extremal mappings between tori have been recently considered by Gehring [7].

38.6. THEOREM. If $n \geq 3$, then $K_I(D) = 1$ (and $K_O(D) = 1$) iff D is a ball or a half space or the exterior of a ball.

Proof. Let $f : D \to B^n$ be a qc mapping such that $K_I(f) = K_I(D)$. By the theorem of Gehring and Rešetnjak (Remark 13.7.2), which is not proved in these notes, $K_I(f) = 1$ iff f is a Möbius transformation. ΔΔ

38.7. Remarks. 1. If $n = 2$, the Riemann mapping theorem implies that $K_I(D) = K_O(D) = 1$ whenever $\complement D$ is a non-degenerate continuum. Thus there is a striking difference between the cases $n = 2$ and $n \geq 3$.

2. The author does not know whether $K_I(D, D') = 1$ implies that D can be mapped onto D' by a conformal mapping.

38.8. THEOREM. Suppose that (D_j) is a sequence of domains in $\bar{R}^n$ and that D is a domain in $\bar{R}^n$ such that the following conditions are satisfied:

(1) ∂D has at least two points.

(2) If (D_{j_k}) is any subsequence of (D_j), then D is a component of $\ker_{k\to\infty} D_{j_k}$.

Then

$$K_I(D) \leq \liminf_{j\to\infty} K_I(D_j), \quad K_O(D) \leq \liminf_{j\to\infty} K_O(D_j).$$

Proof. We prove the first inequality; the proof for K_O is similar. We may assume that $\liminf K_I(D_j) = K < \infty$. Passing to a subsequence, we may assume that $K_I(D_j) \leq K + 1$ for all j, and that there is a point x_o which belongs to D and to every D_j. Choose two distinct points $a, b \in \partial D$ and their neighborhoods U, V such that $\bar{U} \cap \bar{V} = \emptyset$. Passing again to a subsequence we may assume that $U \cap \partial D_j \neq \emptyset \neq V \cap \partial D_j$ for all j. Choose extremal mappings $f_j : B^n \to D_j$ such that $K_O(f_j) = K_I(D_j)$ and $f_j(0) = x_o$. Since $q(\partial D_j) \geq q(U, V) > 0$, it follows from 19.2 and 20.5 that $\{f_j \mid j \in N\}$ is a normal family. Passing once more to a subsequence, we may therefore assume that $f_j \to f$ c-uniformly in R^n. Since $f(0) = x_0 \in \ker D_j$, it follows from 21.9 that f is a homeomorphism onto a component of $\ker D_j$, hence onto D. From 37.2 it follows that $K_I(D) \leq K_O(f) \leq \liminf K_O(f_j) = \liminf K_I(D_j)$. ∆∆

38.9. Example. Let D_j be the cylinder $x_1^2 + x_2^2 < j^2$, $|x_3| < 1$ in R^3, and let D be the domain between the planes $x_3 = \pm 1$. Then the conditions of 38.8 are satisfied. By 17.23 we obtain $\lim K_I(D_j) = \lim K_O(D_j) = \infty$. A direct proof for this is given in Gehring-Väisälä [2, p. 41].

38.10. Definition. A domain $D \subset R^n$ is called raylike with vertex at $v \in \partial D$, $v \neq \infty$, if $v + t(x-v) \in D$ whenever $x \in D$ and $t > 0$. For example, the half space $x_1 > 0$, the quarter space $x_1 > 0$, $x_2 > 0$, etc. are raylike with vertex at the origin.

38.11. THEOREM. Suppose that D and G are domains in R^n such that G is raylike with vertex at v. If v has a neighborhood U such that $U \cap D = U \cap G$, then $K_I(D) \geq K_I(G)$ and $K_O(D) \geq K_O(G)$.

Proof. Define $f_j : R^n \to R^n$ by $f_j(x) = v + j(x-v)$. Then f_j is conformal, and $G = \ker (f_{j_k} D)$ for every subsequence (f_{j_k}). Hence it follows from 38.8 that $K_I(G) \leq \liminf K_I(f_j D) = K_I(D)$, and similarly for the outer coefficient $K_O(G)$. ∆∆

38.12. Example. Let $n \geq 3$, let D be the cube $0 < x_i < 1$, and let G be the quarter space $x_1 > 0$, $x_2 > 0$. Then the conditions of 38.11 are satisfied with $v = e_3/2$. Hence the coefficients of D are not smaller than those of G. We shall give a direct proof for the inner coefficient in Section 40 by showing that $K_I(G) = 2$, $K_I(D) \geq 2$.

38.13. Remark. This section is from Gehring-Väisälä [2, pp. 7-11].

39. Spherical rings

39.1. THEOREM. Let D be the spherical ring $1 < |x| < a$, and let D' be the spherical ring $1 < |x| < b$. If $a \leq b$, then

$$K_I(D, D') = \frac{\log b}{\log a}, \qquad K_O(D, D') = \left(\frac{\log b}{\log a}\right)^{n-1}.$$

Proof. Set $q = \log b / \log a$. Define $f: D \to D'$ by $f(x) = |x|^{q-1}x$. From 16.2 it follows that $K_I(f) = q$ and $K_O(f) = q^{n-1}$. Hence it remains to prove that $K_I(g) \geq q$ and $K_O(g) \geq q^{n-1}$ for every homeomorphism $g: D \to D'$. If $\Gamma = \Delta_0(S(1), S(a), D)$, then $\Gamma' = \Delta_0(S(1), S(b), D')$, and we obtain by 7.5 and 11.3,

$$K_O(g) \geq \frac{M(\Gamma)}{M(\Gamma')} = \frac{\omega_{n-1}(\log a)^{1-n}}{\omega_{n-1}(\log b)^{1-n}} = q^{n-1}.$$

To prove the inequality $K_I(g) \geq q$ we consider the family $\Gamma = \Delta(E, F, D)$ where $E = \{te_n \mid 1 < t < a\}$ and $F = \{te_n \mid -a < t < -1\}$. Then it follows from 10.12 that $M(\Gamma) = c_n \log a$ and $M(\Gamma') \geq c_n \log b$. This implies $K_I(g) \geq M(\Gamma')/M(\Gamma) \geq q$. ΔΔ

39.2. Remark. Contrary to the following sections, 39.1 has sense also for $n = 2$. However, in this case the proof is simpler, because $K_I(D, D') = K_O(D, D')$.

40. Wedges

Throughout this section we assume that $n \geq 3$. We shall use in R^n cylindrical coordinates (r, φ, z), defined in 16.3. As in 16.3, we denote by D_α the domain $\{x \in R^n \mid 0 < \varphi < \alpha\}$, $0 < \alpha \leq 2\pi$.

40.1. Definition. A domain $D \subset R^n$ is a wedge of angle α if D can be mapped onto D_α by a similarity mapping f. The image of the (n-2)-dimensional subspace $r = 0$ under f^{-1} is called the edge of the wedge D. A boundary point b of a domain D is called a wedge point of angle α if there is a neighborhood U of b and a wedge G of angle α such that b is on the edge of G and such that $U \cap G = U \cap D$.

This section deals with mappings between domains which have wedge points. In particular, we calculate the inner coefficient of a convex wedge.

40.2. LEMMA. If b is a wedge point of angle α of a domain D and if $\alpha < 2\pi$, then D is locally qclly bi-collared at b (see 35.4).

Proof. We may assume that $b = 0$ and that $D \cap B^n = D_\alpha \cap B^n$. Define $f : R^n \to R^n$ by $f(r, \varphi, z) = (r, \pi\varphi/\alpha, z)$ for $0 \leq \varphi \leq \alpha$ and $f(r, \varphi, z) = (r, \pi + \pi(\varphi-\alpha)/(2\pi-\alpha), z)$ for $\alpha < \varphi < 2\pi$. By 16.3, f is qc in D_α and in $C\bar{D}_\alpha$. By 35.1, f is qc. Since f maps B^n onto itself and $B^n \cap D_\alpha$ onto $B^n \cap D_\pi$, D is locally qclly bi-collared at 0. ΔΔ

40.3. THEOREM. Suppose that x_0 is a wedge point of angle α of D and that y_0 is a wedge point of angle β of D' such that $0<\alpha\le\beta\le\pi$. If $f: D\to D'$ is a homeomorphism such that $y_0 \in C(f, x_0)$, then $K_I(f)\ge\beta/\alpha$. The bound is best possible.

Proof. We may assume that f is qc. By 35.5 and 40.2 we can find a neighborhood U of x_0, a neighborhood U' of y_0, and a qc mapping $g: U\to U'$ such that $g\mid U\cap D = f\mid U\cap D$ and such that $U\cap D = U\cap G$, $U'\cap D' = U'\cap G'$ where G and G' are wedges of angles α and β, respectively. Let E be the edge of G. Since $n\ge 3$, there is a point $x_1\in E\cap U$ such that $L(x_1, g)>0$, since otherwise g would be locally constant in $E\cap U$. Since $g(x_1)\in g(U\cap\partial D) = U'\cap\partial D'$, $g(x_1)$ is a wedge point of angle α' of D' where α' is either β or π. To simplify notations we assume, as we may, that $x_1 = 0 = g(x_1)$ and that there are neighborhoods V and V' of 0 such that $gV = V'$ and such that $V\cap D = V\cap D_\alpha$, $V'\cap D' = V'\cap D_{\alpha'}$. Since $H(0, g^{-1})<\infty$, there are positive constants H and r such that $|y|\le r$ implies $y\in V'$ and that $L(0, g^{-1}, |y|)\le H\,\ell(0, g^{-1}, y)$. Next choose δ such that $0<\delta<L(0, g)$ and a sequence of points $z_j\in V$ such that $z_j\to 0$ and such that $\delta|z_j|\le |g(z_j)|<r$. Set $a_j = |g(z_j)|$. If $|y| = a_j$, then $|g^{-1}(y)|\le L(0, g^{-1}, a_j)\le H\,\ell(0, g^{-1}, a_j)\le H|z_j|\le Ha_j/\delta$. Set $b = \ell(0, g^{-1}, r)$, and choose $j\in N$ such that $Ha_j/\delta<b$. Let A_j be the spherical ring $\{y \mid a_j<|y|<r\}$, and set $\Gamma_j' = \{\gamma\in\Gamma_{A_j} \mid |\gamma|\subset D'\}$. Then 7.7 implies

$$M(\Gamma_j') = \frac{\alpha'\omega_{n-1}}{2\pi}\left(\log\frac{r}{a_j}\right)^{1-n}.$$

On the other hand, $\Gamma_j = f^{-1}\Gamma_j'$ is minorized by the family $\Delta_j = \{\gamma\in\Gamma_{R_j} \mid |\gamma|\subset D_\alpha\}$ where R_j is the ring $Ha_j/\delta<|x|<b$. Hence

$$M(\Gamma_j)\le M(\Delta_j) = \frac{\alpha\,\omega_{n-1}}{2\pi}\left(\log\frac{b\delta}{Ha_j}\right)^{1-n}.$$

Consequently,

$$K_I(f) \geq \frac{M(\Gamma_j')}{M(\Gamma_j)} \geq \frac{\alpha'}{\alpha}\left(\frac{\log \frac{b\delta}{Ha_j}}{\log \frac{r}{a_j}}\right)^{n-1} .$$

Letting $j \to \infty$ yields $K_I(f) \geq \alpha'/\alpha \geq \beta/\alpha$.

In 16.3 we showed that the folding $f : D_\alpha \to D_\beta$ has inner dilatation β/α. Hence the bound $K_I(f) \geq \beta/\alpha$ is sharp. ⩟

40.4. <u>THEOREM.</u> If $0 < \alpha \leq \beta \leq \pi$, then $K_I(D_\alpha, D_\beta) = \beta/\alpha$. In particular, $K_I(D_\alpha) = \pi/\alpha$ for $0 < \alpha \leq \pi$.

<u>Proof.</u> Suppose that $f : D_\alpha \to D_\beta$ is a qc mapping. Choose a point x_0 on the edge of D_α and a point $y_0 \in C(f, x_0)$ such that $y_0 \neq \infty$. Then y_0 is a wedge point of angle α' of D_β, where α' is either β or π. Hence 40.3 implies $K_I(f) \geq \alpha'/\alpha \geq \beta/\alpha$. This proves $K_I(D_\alpha, D_\beta) \geq \beta/\alpha$. On the other hand, since the folding of D_α onto D_β has inner dilatation β/α, $K_I(D_\alpha, D_\beta) \leq \beta/\alpha$. ⩟

40.5. <u>Remarks.</u> 1. The value of $K_O(D_\alpha)$ is not known for $0 < \alpha < \pi$. However, it is easy to show that $(\pi/\alpha)^{1/(n-1)} \leq K_O(D_\alpha) \leq \pi/\alpha$. The left-hand inequality follows from (38.3). Setting $f(r, \varphi, z) = (r, \pi\varphi/\alpha, \pi z/\alpha)$ we obtain a qc mapping $f : D_\alpha \to D_\pi$ for which $K_O(\alpha) = \pi/\alpha$. This proves the right-hand inequality.

2. If $\pi < \alpha \leq 2\pi$, then both coefficients of D_α are unknown.

3. Theorem 40.3 can easily be generalized to <u>curvilinear wedges</u>. A domain D is said to have a curvilinear wedge of angle α at $x_0 \in \partial D$ if for each $K > 1$ there is a neighborhood U of x_0 and a K-qc mapping $f : U \to B^n$ such that $f(U \cap D) = B^n \cap D_\alpha$. For example, if D is an ordinary right cylinder in R^3, we have $K_I(D) \geq 2$.

4. One has been able to calculate the coefficients of qcty for only very few domains. In addition to the convex wedge, one knows

the outer coefficient of an infinite cylinder and an infinite convex cone (Gehring-Väisälä [2]), and the inner coefficient of an infinite non-convex cone (Vamanamurthy [1]). Furthermore, one can show that suitable deformations of these domains do not change the coefficient. For example, if $\alpha < \pi$ and if $D = D_\alpha \cup (D_\beta + e_1)$, then $K_I(D) = K_I(D_\alpha) = \pi/\alpha$ if β is slightly greater than α.

5. The result $K_I(D_\alpha) = \pi/\alpha$ was first proved by Gehring and Väisälä [2], by a different method. The above proof is due to Gehring, and it is based on an idea of Syčev [1].

41. Jordan domains

As noted in Section 17, a Jordan domain D in $\bar{R}^n$ need not be homeomorphic to B^n, and even if so, it need not be qclly equivalent to B^n. However, it is known that D is homeomorphic to B^n if it has a collared boundary. By this we mean that there are neighborhoods U of ∂D and V of ∂B^n and a homeomorphism $f: U \cap \bar{D} \to V \cap \bar{B}^n$. If, in addition, f is qc in $U \cap D$, we say that D has a <u>quasiconformally collared boundary</u>. The main result of this section states that if D has a qclly collared boundary, then D is qclly equivalent to a ball.

41.1. <u>LEMMA.</u> Suppose that

(1) D_1 and D_2 are Jordan domains such that $\bar{D}_1 \cap \bar{D}_2 = \emptyset$ and $\bar{D}_1 \cup \bar{D}_2 \subset B^n$.

(2) B_1 and B_2 are open balls such that $\bar{B}_1 \cap \bar{B}_2 = \emptyset$ and $\bar{B}_1 \cup \bar{B}_2 \subset B^n$.

(3) f is a homeomorphism of $\underline{C}(D_1 \cup D_2)$ onto $\underline{C}(B_1 \cup B_2)$ such that $f\partial D_i = \partial B_i$, and $f \mid \underline{C}(\bar{D}_1 \cup \bar{D}_2)$ is qc.

(4) $f(x) = x$ in a neighborhood of $\underline{C}B^n$.

Then there exists a homeomorphism $f^* : \underline{C}D_2 \to \underline{C}B_2$ such that $f^* | \partial D_2 = f | \partial D_2$ and such that $f^* | \underline{C}\bar{D}_2$ is qc.

Proof. We may assume that there are numbers $-1 < a < b < 1$ such that B_1 and B_2 lie in the half spaces $x_n < a$ and $x_n > b$, respectively. Set

$$E = \bigcup_{j=0}^{\infty} (D_1 \cup D_2 + 3je_1), \quad E' = \bigcup_{j=0}^{\infty} (B_1 \cup B_2 + 3je_1),$$

and define $g : \underline{C}E \to \underline{C}E'$ by $g(x) = f(x - 3je_1) + 3je_1$ if $x \in (\bar{B}^n + 3je_1) \setminus E$ for some $j \in N \cup \{0\}$, and by $g(x) = x$ otherwise. Then g is obviously a homeomorphism. Moreover, every point x in $\underline{C}\bar{E}$ has a neighborhood U such that $g | U$ is composed of f and translations. Hence $K(g | \underline{C}\bar{E}) \le K(f)$. Set $r = (b-a)/3$, and define $k : R^1 \to R^1$ by

$$\begin{aligned} k(t) &= 0 && \text{for } t \le a + r, \\ k(t) &= (a+r-t)/r && \text{for } a + r < t \le b - r, \\ k(t) &= -1 && \text{for } t > b - r. \end{aligned}$$

Setting $h(x) = x + 3k(x_n)e_1$ we obtain a homeomorphism $h : R^n \to R^n$. Since h is piecewise linear, it follows from 35.1 that h is qc. We show that setting

$$\begin{aligned} f^*(x) &= g^{-1}(h(g(x))) + 3e_1 && \text{for } x \in R^n \setminus E, \\ f^*(x) &= x + 3e_1 && \text{for } x \in D_1 + 3je_1, \ j \ge 0, \\ f^*(x) &= x && \text{for } x \in D_2 + 3je_1, \ j \ge 1, \\ f^*(\infty) &= \infty, \end{aligned}$$

we obtain the required mapping.

It is clear that f^* is bijective and that f^* is continuous except possibly in the sets $\partial D_i + 3je_1$ and at ∞. Consider, for example, a point $x_0 \in \partial D_1 + 3je_1$. Then x_0 has a neighborhood U such that $g(U \setminus E) \subset (B^n + 3je_1) \cap \{x \mid x_n < a + r\}$. For $x \in U \setminus E$ we have $g^{-1}(h(g(x))) = x$. Hence f^* is continuous at x_0 . Furthermore, it follows from the definition of f^* that $|f^*(x) - x| < 5$ for all finite x . Hence f^* is continuous at ∞ . Thus f^* is a homeomorphism.

If $x \in \partial D_2$, then $f^*(x) = g^{-1}(h(g(x))) + 3e_1 = g^{-1}(f(x) - 3e_1) + 3e_1 = f(x)$. Finally, using 34.7 and 17.3 we see that $K(f^* | \underline{C}\bar{D}_2) \leq K(g)^2 K(h) < \infty$. ΔΔ

41.2. <u>LEMMA.</u> In addition to (1), (2), and (3) of 41.1, suppose that

(4') $0 \in D_2$.

(5') $\underline{C} f \underline{C} \bar{B}^n \subset B^n$.

Then there exists a homeomorphism $f^* : \underline{C}D_2 \to \underline{C}B_2$ with the same properties as in 41.1.

<u>Proof.</u> Choose $0 < a < b < 1$ such that $B^n(a) \subset D_2$ and $D_1 \cup D_2 \subset B^n(b)$. Set $q = \log b / \log a$, and define $g : \bar{R}^n \to \bar{R}^n$ by $g(x) = |x|^{q-1} x$ for $x \in B^n$ and $g(x) = x$ for $x \in \underline{C}B^n$. By 16.2 and 35.1, g is qc. Set $D_1' = fgD_1 = \underline{C} fg \underline{C} D_1$ and $D_2' = \underline{C} fg \underline{C} D_2$. Then $f \circ g$ maps $\underline{C}D_2$ onto $\underline{C}D_2'$, and $h = f \circ (f \circ g)^{-1}$ maps $\underline{C}(D_1' \cup D_2')$ onto $\underline{C}(B_1 \cup B_2)$. Obviously, h satisfies the conditions (1), (2), (3) of 41.1, with D_i replaced by D_i' . Moreover, $f\underline{C}\bar{B}^n$ is a neighborhood of $\underline{C}B^n$, and if $x \in f\underline{C}\bar{B}^n$, then $h(x) = f(g^{-1}(f^{-1}(x))) = x$. Thus h satisfies also the condition (4) of 41.1. Consequently, there is a homeomorphism $h^* : \underline{C}D_2' \to \underline{C}B_2$ such that $h^* | \partial D_2' = h | \partial D_2'$ and such that $h^* | \underline{C}\bar{D}_2'$ is qc. The required mapping f^* is obtained by setting $f^*(x) = h^*(f(g(x)))$. In fact, if $x \in \partial D_2$, then $f(g(x)) \in \partial D_2'$, and $f^*(x) = h(f(g(x))) = f(x)$. ΔΔ

41.3. <u>THEOREM.</u> Suppose that f is a homeomorphism of $E = \{x \in R^n \mid a < |x| \leq 1\}$ onto a set $E' \subset \bar{R}^n$ such that $f | \operatorname{int} E$ is qc.

Let D be the component of $\underline{C}fS^{n-1}$ whose closure contains E'. Then there exists a homeomorphism $f^*: \bar{B}^n \to \bar{D}$ such that $f^* | S^{n-1} = f | S^{n-1}$ and such that $f^* | B^n$ is qc.

Proof. Choose $a < b < 1$. The image of $\{x \mid b \le |x| \le 1\}$ under f can be written as $\underline{C}(D_1 \cup D_2)$ where D_1 and D_2 are Jordan domains, $\partial D_1 = fS^{n-1}(b)$ and $\partial D_2 = fS^{n-1} = \partial D$. Choose a Möbius transformation g such that $g\bar{D}_1 \subset B^n$, $g\bar{D}_2 \subset B^n$ and $0 \in gD_2$. Next choose a Möbius transformation h such that $\underline{C}hf^{-1}g^{-1}\underline{C}\bar{B}^n \subset B^n$. Then $f_1 = h \circ f^{-1} \circ g^{-1}$: $\underline{C}(gD_1 \cup gD_2) \to \underline{C}(hB^n(b) \cup h\underline{C}\bar{B}^n)$ is a homeomorphism which satisfies the conditions of 41.2. Hence there is a homeomorphism $f_1^* : \underline{C}gD_2 \to h\bar{B}^n$ such that $f_1^*(x) = f_1(x)$ for $x \in \partial gD_2$ and such that $f_1^* | \underline{C}g\bar{D}_2$ is qc. The required mapping is then $f^* = g^{-1} \circ f_1^{*-1} \circ h : \bar{B}^n \to \bar{D}$. ∆∆

41.4. COROLLARY. If a Jordan domain D has a qclly collared boundary, then D is qclly equivalent to a ball. ∆∆

We next give a slightly strengthened version of 41.3. Let A be the ring $a < |x| < 1$, and let $f : A \to A'$ be a qc mapping. Observe that f need not be defined on S^{n-1}. Then A' is a ring $R(C_0, C_1)$, and we choose the notation so that $C(f, S^{n-1}) \subset C_1$. We show that the domain $D = \underline{C}C_1$ is qclly equivalent to B^n. Fix $a < b < 1$. Applying 41.3 we find a homeomorphism f^* of $\bar{B}^n(b)$ onto the closure of the component of $\underline{C}fS^{n-1}(b)$ which contains C_0, such that $f^* | S^{n-1}(b) = f | S^{n-1}(b)$ and such that $f^* | B^n(b)$ is qc. Defining $f^*(x) = f(x)$ for $b < |x| < 1$ we obtain a homeomorphism $f^* : B^n \to D$. By 35.1, f^* is qc. We have thus proved:

41.5. THEOREM. Suppose that f is a qc mapping of the spherical ring $a < |x| < 1$ onto a ring A'. Denote by C_1 the component of $\underline{C}A'$ which contains the cluster set $C(f, S^{n-1})$. Then for each

$b \in (a, 1)$ there is a qc mapping $f^* : B^n \to \underline{C}C_1$ such that $f^*(x) = f(x)$ for $b \leq |x| < 1$. ∆∆

In particular, the mapping f^* of 41.3 can be chosen so that $f^*(x) = f(x)$ for $b \leq |x| \leq 1$.

We finally give an application of 41.3.

41.6. <u>THEOREM.</u> Suppose that $f : B^n \to D$ is a qc mapping. Then for every compact set $F \subset B^n$ there is a qc mapping $f^* : \bar{R}^n \to \bar{R}^n$ such that $f^* | F = f | F$.

<u>Proof.</u> Choose $b < 1$ such that $F \subset B^n(b)$. Applying 41.3 to $f \circ g$, where g is a suitable inversion, we can find a homeomorphism $f^* : \underline{C}B^n(b) \to \underline{C}fB^n(b)$ such that $f^*(x) = f(x)$ for $|x| = b$ and such that $f^* | \underline{C}\bar{B}^n(b)$ is qc. Defining $f^*(x) = f(x)$ for $|x| < b$ we obtain the required mapping. ∆∆

41.7. <u>Remarks.</u> 1. This section is from Gehring [6]. The topological form of 41.4 was proved in 1959 by Mazur [1] under an additional niceness condition. The proof of 41.1 is an explicit version of Mazur's proof. In 1960 Morse [2] showed how to get rid of this superfluous condition. The proof of 41.2 is an explicit version of Morse's proof. A different proof for the topological case was given simultaneously by Brown [1]. His proof, however, makes use of non-qc homeomorphisms. The proof given in these notes applies also to the topological case, because the qcty of f was used only to conclude the qcty of f.

2. From 41.4 it follows that a Jordan domain with smooth boundary is qclly equivalent to a ball. Indeed, suppose that g is a diffeomorphism of S^{n-1} onto a differentiable submanifold of R^n. Then it can be shown (Morse [1]) that g can be extended to a diffeomorphism, and hence to a qc mapping, of a neighborhood of S^{n-1}.

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A list of special symbols is given on pages X-XI.