Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1319

Matti Vuorinen

Conformal Geometry and Quasiregular Mappings

Springer-Verlag
Berlin Heidelberg New York London Paris Tokyo

Author

Matti Vuorinen
Department of Mathematics, University of Helsinki
Hallitusk. 15, 00100 Helsinki, Finland

Mathematics Subject Classification (1980): 30C60

ISBN 3-540-19342-1 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-19342-1 Springer-Verlag New York Berlin Heidelberg

Printed in Germany

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210

Contents

Preface

This book is based on my lectures on quasiregular mappings in the euclidean n-space $\mathbf{R}^n$ given at the University of Helsinki in 1986. It is assumed that the reader is familiar with basic real analysis or with some basic facts about quasiconformal mappings (an excellent reference is pp. 1–50 in J. Väisälä's book [V7]), but otherwise I have tried to make the text as self-contained and easily accessible as possible. For the reader's convenience and for the sake of easy reference I have included without proof most of those results from [V7] which will be exploited here. I have also included a brief review of those properties of Möbius transformations in $\mathbf{R}^n$ which will be used throughout.

In order to make the text more useful for students I have included nearly a hundred exercises, which are scattered throughout the book. They are of varying difficulty, with hints for solution provided for some. For specialists in the field I have included a list of open problems at the end of the book. The bibliography contains, besides references, additional items which are closely related to the subject matter of this book.

From its beginning twenty years ago the subject of quasiregular mappings in n-space has developed into an extensive mathematical theory having connections with PDE theory, calculus of variations, non-linear potential theory, and especially geometric function theory and quasiconformal mapping theory. Excellent contributions to this subject have been made, in particular, by the following five mathematicians:

F. W. Gehring, O. Martio, Yu. G. Reshetnyak, S. Rickman, and J. Väisälä.

The subject matter of this book relies heavily on their work. I am indebted to them not only for their scientific contributions but also for the help and advice they have given me during the various stages of my work. It was O. Martio who suggested I start writing this book. The writing was made possible by a research fellowship of the Academy of Finland, which I held in 1979–85. A draft for the text was finished in the fall of 1982 during my stay at the Mittag-Leffler Institute in Sweden.

The following mathematicians have provided their generous help by checking various versions of the manuscript, pointing out errors, and contributing corrections: J. Heinonen, G. D. Anderson, and M. K. Vamanamurthy. Useful remarks were also made by J. Ferrand and P. Järvi. At the final stage I have had the good fortune to work with J. Kankaanpää, who prepared the final version of the text using the TEX system of D. E. Knuth and improved the text in various ways. The previewer program for TEX written by A. Hohti was very helpful in the course of this project. The work of Kankaanpää was supported by a grant of the Academy of Finland. Hohti and O. Kanerva have provided their generous assistance in the use of the TEX system.

Helsinki
October 1987
Matti Vuorinen

Introduction

Quasiconformal and quasiregular mappings in $\mathbf{R}^n$ are natural generalizations of conformal and analytic functions of one complex variable, respectively. In the two-dimensional case these mappings were introduced by H. Grötzsch [GRÖ] in 1928 and the higher-dimensional case was first studied by M. A. Lavrent'ev [LAV] in 1938. Far-reaching results were obtained also by O. Teichmüller [TE] and L. V. Ahlfors [A1]. The systematic study of quasiconformal mappings in $\mathbf{R}^n$ was begun by F. W. Gehring [G1] and J. Väisälä [V1] in 1961, and the study of quasiregular mappings by Yu. G. Reshetnyak in 1966 [R1]. In a highly significant series of papers published in 1966–69 Reshetnyak proved the fundamental properties of quasiregular mappings by exploiting tools from differential geometry, non-linear PDE theory, and the theory of Sobolev spaces.

In 1969–72 O. Martio, S. Rickman and J. Väisälä ([MRV1]–[MRV3], [V8]) gave a second approach to the theory of quasiregular mappings which was based on some results of Reshetnyak, most notably on the fact that a non-constant quasiregular mapping is discrete and open. On the other hand, their approach made use of tools from the theory of quasiconformal mappings, such as curve families and moduli of curve families. The extremal length and modulus of a curve family were introduced by L. V. Ahlfors and A. Beurling in their celebrated paper [AB] on conformal invariants in 1950.

A third approach was suggested by B. Bojarski and T. Iwaniec [BI2] in 1983. Their methods are real analytic in nature and largely independent of Reshetnyak's work.

In this book a fourth approach is suggested, which is a ramification of the curve family method in [MRV1]–[MRV3] and in which conformal invariants play a central role. Each of the above three approaches yields a theory covering the whole spectrum of results of the theory of quasiregular mappings. So far the fourth approach of this book, introduced by the author in [VU10]–[VU13] has been applied mainly to distortion theory. This work has been continued in [AVV1], [AVV2], [FV], [LEVU], where some quantitative distortion theorems were discovered. These papers also include results which are sharp as the maximal dilatation K approaches 1. Perhaps surprisingly it also turned out in [AVV1] that to a considerable degree a distortion theory can be developed independently of the dimension n.

In short, this fourth approach consists of the following. In a domain G in $\mathbf{R}^n$ one studies two conformal invariants $\lambda_G(x,y)$ and $\mu_G(x,y)$ associated with a pair of points x and y in G. These invariants were apparently first introduced by J. Ferrand [LF2] in 1973 and I. S. Gál [GÁL] in 1960, respectively. The systematic application of these invariants was begun by the author in a recent series of papers [VU10]–[VU13]. By their definitions, $\lambda_G(x,y)$ and $\mu_G(x,y)$ are solutions of certain extremal problems associated with the moduli of some curve families. To derive distortion theorems exploiting λ_G and μ_G we require two things:

(a) the quasiinvariance of moduli of curve families under quasiconformal and quasiregular mappings ([MRV1]–[MRV3]),

(b) quantitative estimates for λ_G and μ_G in terms of "geometric quantities".

For a general domain G in $\mathbf{R}^n$ these invariants have no explicit expression. In the particular case $G = \mathbf{B}^n$ such an expression is known for both λ_G and μ_G, and for $G = \mathbf{R}^n \setminus \{0\}$ good two–sided estimates for the invariant λ_G will be obtained. We then generalize these results for a wider class of domains. In the two–dimensional case we can obtain the exact value of $\lambda_{\mathbf{R}^2\setminus\{0\}}(x,y)$ if we use the solution of a classical extremal problem of geometric function theory, the modulus problem of O. Teichmüller [KU, Ch. V].

This book is divided into four chapters. Chapter I deals with geometric preliminaries, including a discussion of Möbius transformations. In Chapter II we study certain conformal invariants and apply these results in Chapter III to obtain distortion theorems, the main theme of this book. The final part, Chapter IV, is a brief discussion of some boundary properties of quasiconformal mappings.

A survey of quasiregular mappings

The goal of this survey is to give the reader a brief overview of the theory of quasiconformal (qc) and quasiregular (qr) mappings and of some related topics. We shall also try to indicate the many ways in which the classical function theory of one complex variable (CFT) is related to quasiregular mapping theory (QRT) in $\mathbf{R}^n$ as well as to point out some differences between CFT and QRT. This survey deals chiefly with results not discussed elsewhere in the book.

For a general orientation the reader is urged to read some of the existing excellent surveys [A4], [L1], [L2], [BAM], [G4], [G8]–[G10], [I], and [V10], of which the first three deal with the two-dimensional case and the others the multidimensional case. Several open problems are listed in the surveys of A. Baernstein and J. Manfredi [BAM], F. W. Gehring [G9], and J. Väisälä [V10].

1. Foundations. In his pioneering papers [R1]–[R10], in which were laid the foundations of QRT, Yu. G. Reshetnyak successfully combined the powerful analytic machinery of PDE's in the sense of Sobolev with some geometric ideas from CFT. Reshetnyak showed that the basic properties of qr mappings can be derived from the properties of the function $u_f(x) = \log|f(x)|$, where f is qr. He proved that u_f satisfies a non-linear elliptic PDE which for $n = 2$ is linear and coincides with the Laplace equation. It follows from the work of J. Moser [MOS], F. John – L. Nirenberg, and J. Serrin [SE] that the solutions of this equation satisfy the Harnack inequality in $\{ z : u_f(z) > 0 \}$. Note that if f is analytic, then $\log|f(z)|$ has a similar role in CFT. Obviously only a part of CFT can be carried over to QRT: for instance power series expansions and the Riemann mapping theorem have no n-dimensional counterpart.

2. Quasiconformal balls. By Riemann's mapping theorem a simply-connected plane domain with more than one boundary point can be mapped conformally onto the unit disk $\mathbf{B}^2$. Liouville's theorem says that the only conformal mappings in $\mathbf{R}^n$, $n \geq 3$, are the Möbius transformations. Thus Riemann's mapping theorem has no counterpart in $\mathbf{R}^n$ when $n \geq 3$: since Möbius transformations preserve spheres, the unit ball $\mathbf{B}^n$ in $\mathbf{R}^n$ can be mapped conformally only onto another ball or a half–space. A quasiconformal counterpart of the Riemann mapping theorem is also false: for $n \geq 3$ there are Jordan domains in $\mathbf{R}^n$ homeomorphic to $\mathbf{B}^n$ which cannot be mapped quasiconformally onto $\mathbf{B}^n$ although their complements can be so mapped. Also, the unit ball $\mathbf{B}^n$, $n \geq 3$, can be mapped quasiconformally onto a domain with non–accessible boundary points, as shown by Gehring and Väisälä in [GV1]. This fact shows that for each $n \geq 3$ the quasiconformal mappings in $\mathbf{R}^n$ constitute a class of mappings substantially larger than the class of Möbius transformations.

3. Topological properties. A basic fact from CFT is that a non–constant analytic function is discrete (i.e. point–inverses $f^{-1}(y)$ are discrete sets if f analytic) and open (i.e. fA is open whenever f is analytic and A is open). By Reshetnyak's fundamental work a similar result holds in QRT. Next let B_f denote the set of all points where f fails to be a local homeomorphism. In CFT it is a basic fact that B_f is a discrete set if f is non–constant and analytic. A topological difference between the cases $n = 2$ and $n \geq 3$ is that B_f is never discrete if f is qr in $\mathbf{R}^n$, $n \geq 3$, and $B_f \neq \emptyset$. By a result of A. V. Chernavskiĭ $\dim B_f = \dim fB_f \leq n-2$ if $f\colon G \to \mathbf{R}^n$ (G a domain in $\mathbf{R}^n$) is discrete and open ([CHE1], [CHE2], [V5]). Also the metric properties are different: if $n = 2$ and f is analytic, then $\operatorname{cap} B_f = 0$, while if $n \geq 3$ and f is qr in $\mathbf{R}^n$, then either $B_f = \emptyset$ or $\operatorname{cap} B_f > 0$ (for the definition of the capacity see Section 7; see also [R10], [MR2], [S2]).

By a result of S. Stoilow a qr mapping f of $\mathbf{B}^2$ onto a domain D can be represented as $f = g \circ h$, where h is a qc mapping of $\mathbf{B}^2$ onto itself and g is an analytic function ([LV2]). Thus the powerful two–dimensional arsenal of CFT is applicable to the "analytic part" of f, greatly facilitating the study of two–dimensional qr mappings. No such result is known for the multidimensional case.

Another result which is known only for the dimension $n = 2$ is the powerful existence theorem for plane quasiconformal mappings (cf. [LV2]). In the multidimensional case there is no general existence theorem and all examples of qc and qr mappings known to the author are based on direct constructions. In the qc case several examples are given in [GV1]. In the qr case a basic mapping is the winding mapping, given in the cylindrical coordinates (r, φ, z) by $(r, \varphi, z) \mapsto (r, k\varphi, z)$, k a positive integer [MRV1]. An important example of a qr mapping is the so called Zorich mapping ([ZO1], [MSR1]) and its various generalizations due to Rickman (cf. e.g. [RI1l]).

Additional examples are given in [R12, pp. 27–32], [MSR2], and [MSR3]. One can also construct new qc (qr) mappings by composing qc (qr) mappings.

4. Quasiconformality versus Lipschitz and Hölder maps. A homeomorphism $f: G \to fG$, $G \subset \mathbf{R}^n$, is said to be K–qc if

$$(*) \qquad \mathsf{M}(\Gamma)/K \le \mathsf{M}(f\Gamma) \le K\,\mathsf{M}(\Gamma)$$

for all curve families Γ in G where $\mathsf{M}(\Gamma)$ is the modulus of Γ (see Section 5 below). This definition is somewhat implicit because the concept of modulus is rather complicated. To clarify the geometric consequences of $(*)$ let us point out that

$$H(x,f) = \limsup_{r\to 0}\left\{ \frac{|f(x)-f(z)|}{|f(x)-f(y)|} : |z-x| = r = |y-x| \right\} \le d(n,K)$$

for all $x \in G$, where $d(n,K) < \infty$ depends only on n and K. A well-known property of conformal mappings can be expressed by stating that $H(x,f) = 1$ for $K = 1$ (while, unfortunately, $d(n,K) \not\to 1$ as $K \to 1$ for $n \ge 3$, cf. p. 193).

A homeomorphism $f: G \to fG$ satisfying

$$|x-y|/L \le |f(x)-f(y)| \le L|x-y|$$

for all $x, y \in G$, is called L–bilipschitz. It is easy to show that L–bilipschitz maps are $L^{2(n-1)}$–qc. But the converse is false. The standard counterexample is the qc radial stretching $x \mapsto |x|^{\alpha-1}x$, $x \in \mathbf{B}^n$, $\alpha \in (0,1)$, which is not bilipschitz. All qc mappings are, however, locally Hölder continuous; e.g., if $f: \mathbf{B}^n \to \mathbf{B}^n$ is K–qc, then for $|x|, |y| \le \frac{1}{2}$

$$|f(x)-f(y)| \le A(n,K)\,|x-y|^{\alpha}\,,\ \alpha = K^{1/(1-n)}\,,$$

where $A(n,K)$ depends only on n and K. For details see Section 11 below.

Let $\mathcal{B}$, $\mathcal{QC}$, and $\mathcal{H}$ denote the classes of all bilipschitz, qc, and locally Hölder continuous mappings. By what was said above the inclusions $\mathcal{B} \subset \mathcal{QC} \subset \mathcal{H}$ hold, where the first inclusion is strict. Simple examples can be constructed to show that also the second inclusion is strict.

Many fundamental features of qc mappings are related to the strictness of the inclusion $\mathcal{B} \subset \mathcal{QC}$. For instance, one can construct qc mappings such that the image of a segment is not even locally rectifiable and such that the Hausdorff dimension of a set is different from the Hausdorff dimension of its image ([GV2]).

The Hölder continuity of qc mappings on the boundary of the domain of definition has been thoroughly investigated by R. Näkki and B. Palka in a series of papers (see e.g. [NP]).

5. L^p–integrability. A K–qc mapping has the property that its partial derivatives are locally L^n–integrable. Moreover, these partial derivatives are even locally L^p–integrable for some $p = p(n, K) > n$. This was proved by B. Bojarski for $n = 2$ and generalized to the multidimensional case by F. W. Gehring [G5]. The method of proof in [G5], which makes use of so–called reverse Hölder inequalities, has found several applications to the calculus of variations and to PDE theory ([GIA], [STR1], [STR2]). Some estimates dealing with the case $K \to 1$ were given by Yu. G. Reshetnyak in [R13] (see also [GUR]). In connection with qr mappings the integrability has also been discussed by B. Bojarski and T. Iwaniec [BI2] and O. Martio [M2].

6. Stability theory. The stability theory of K–qc and K–qr mappings in $\mathbf{R}^n$ in the sense of this book deals with the quantitative description of the behavior of these mappings when $K \to 1$. Roughly speaking, the expectation is that the mapping should become more or less like a conformal mapping under this passage to the limit. By Liouville's classical theorem the two cases $n \geq 3$ and $n \geq 2$ are substantially different, and we shall therefore consider them separately.

Case A. $n \geq 3$. Liouville's classical theorem, which was mentioned above in connection with quasiconformal balls, requires that the mappings be sufficiently smooth (C^3 is enough). By deep results of F. W. Gehring [G2] and Yu. G. Reshetnyak ([R3], R13]) the differentiability assumption can be replaced by the requirement that the mapping be 1–qc or even 1–qr. Recently a different proof was given by B. Bojarski and T. Iwaniec [BI1]. Next, as shown by Reshetnyak ([R3], [R11], [R13]), one can show that as $K \to 1$ any K–qr mapping must approach a Möbius transformation. For the exact statement of these results the reader is referred to [R13]. The methods of [R13] involve normal family arguments. Unfortunately the "speed" with which the convergence to Möbius transformations takes place as $K \to 1$ is usually only qualitatively defined and no quantitative estimate for the "speed" in terms of K and n are known. Additional results have been proved by A. P. Kopylov [KO], J. Sarvas [S3], V. I. Semenov [SEM1], D. A. Trotsenko [TR], and others.

Case B. $n \geq 2$. The paucity of such distortion theorems for K–qc or K–qr mappings in $\mathbf{R}^n$, which are asymptotically sharp as $K \to 1$ and provide quantitative distortion estimates, may be startling when compared to the rich qualitative theory described above in Case A. This state of affairs is due partly to the fact that to prove such results one needs to find sharp estimates for certain little–known special functions. Several results with explicit bounds dealing with the case $K \to 1$ have been proved by V. I. Semenov in several papers (e.g. [SEM1], [SEM2]). Some other distortion theorems of this kind together with associated estimates of special functions were developed in [VU10], [VU11], [AVV1]–[AVV3], [FV].

A survey including some two–dimensional results of this kind is given in [HEL]. See also the important paper [AG] of S. Agard.

7. Dirichlet integral minimizing property. Let G be a domain in $\mathbf{R}^2$ and $v\colon G \to \mathbf{R}$ harmonic. For a domain $D \subset G$ with $\overline{D} \subset G$ let

$$\mathcal{F}_v(D) = \{\, u\colon G \to \mathbf{R} : u|\partial D = v|\partial D\,,\ u \in C^2(G) \,\}\,.$$

A well–known extremal property of the class of harmonic functions, the Dirichlet principle, states that they minimize the Dirichlet integral [T, pp. 9–14]. In the above notation this means that

$$\int_D |\nabla v|^2 dm = \inf_{u \in \mathcal{F}_v(D)} \int_D |\nabla u|^2 dm\,.$$

Analogous Dirichlet integral minimizing properties hold as well for the solutions of the non–linear elliptic PDE's which arise in connection with qr mappings. This important fact was proved by Yu. G. Reshetnyak [R5]. In [MIK3] V. M. Miklyukov continued this research and studied subsolutions of these PDE's.

In a series of papers S. Granlund, P. Lindqvist, and O. Martio have considerably extended these results ([GLM1]–[GLM3], [LI1], [LIM], [M6]). They have also found a unified approach to some function–theoretic parts of QRT including, in particular, the harmonic measure. See also [HMA]. Further results were obtained by J. Heinonen and T. Kilpeläinen.

8. Value distribution theory. In 1967 V. A. Zorich [ZO1] asked whether Picard's theorem holds for spatial qr mappings and whether the value distribution theory of Nevanlinna [NE] has a counterpart in this context. These questions have been answered by S. Rickman in a series of papers [RI3]–[RI11], the main results being reviewed in [RI6] and [RI9]. Additional results appear in [MATR] as well as in [PE1]. An analogue of Picard's theorem was published in [RI4]. One of the methods used in [RI4] is a two–constants theorem for qr mappings (analogous to the two–constants theorem of CFT [NE]), which Rickman derives from an estimate for the solutions of certain non–linear elliptic PDE's due to V. G. Maz'ya [MAZ1]. An alternative proof which only makes use of curve family methods is given in [RI9].

9. Special classes of domains. The standard domain, in which most of the CFT is developed, is the unit disk. During the past ten years an increasing number of papers have been published in which function-theory on a more general domain arises in a natural way. In the early 1960's two highly significant studies of this kind appeared in quite different contexts authored by L. V. Ahlfors and F. John, respectively. Ahlfors studied domains bounded by quasicircles, i. e. images of the usual circle under a qc mapping of $\mathbf{R}^2$, and found remarkable properties of these domains. In a paper related to elasticity properties of materials John introduced a class of domains, nowadays known as John domains.

The importance of John domains was pointed out by Yu. G. Reshetnyak [R11] in connection with injectivity studies of qr mappings. This direction of research was then continued by O. Martio and J. Sarvas [MS2], who also introduced the important class of uniform domains. Uniform domains have found applications in the study of extension operators of function spaces, e. g. in P. Jones' work ([J1], [J2]) as well as elsewhere ([GO], [GM1], [TR], [V12]). Other related classes of domains are QED domains [GM1] and φ-uniform domains ([VU10], [HVU]). The interrelation between some of these classes of domains has been studied by F. W. Gehring in [G8] and [G10], where also several characterizations of quasidisks are given.

Important results dealing with function spaces and their extension to a larger domain have been proved by S. K. Vodop'yanov, V. M. Gol'dstein, and Yu. G. Reshetnyak in [VGR], where additional references can be found.

10. Concluding remarks. The above remarks cover only a part of the existing QRT, and a wider overview can be obtained from the surveys of A. Baernstein and J. Manfredi [BAM] and F. W. Gehring [G9]. We shall conclude this survey by mentioning some directions of active research close to QRT.

Recently qc and qr mappings have appeared in stochastic analysis in B. Øksendal's work [ØK1] and in the theory of manifolds (M. Gromov [GROM]). P. Pansu [PA] has studied quasiconformality in connection with Heisenberg groups, in which he has exploited among other methods the conformal invariant λ_G of J. Ferrand [LF2]. Qc mappings also arise in a natural way in the study of BMO functions (H. M. Reimann–T. Rychener [REIR], K. Astala–F. W. Gehring [ASTG], M. Zinsmeister [ZI]).

In a series of papers V. M. Miklyukov [MIK4] has shown how the extremal length method can be used to study minimal surfaces. Extremely important are the partly topological results connecting geometric topology and quasiconformality, which were proved by D. Sullivan, P. Tukia, J. Väisälä, J. Luukkainen, and others. Discrete groups and quasiconformality have been studied in an important series of papers by P. Tukia ([TU1], [TU2]) and B. N. Apanasov, O. Martio and U. Srebro ([MSR1]–[MSR3]), F. W. Gehring and G. Martin [GMA]. Let us point out that we have confined ourselves here (and also elsewhere in this book) to the case of n-space, $n \geq 2$. For $n = 2$ the reader may consult the excellent surveys of O. Lehto [L1] and [L2] as well as his new book [L3]. The standard references for $n = 2$ are the books by L. V. Ahlfors [A2], H. P. Künzi [KÜ], and O. Lehto and K. I. Virtanen [LV2].

The variety of these results indicates the many ways in which qc and qr mappings arise in mathematics. Many fascinating connections between QRT and other parts of mathematics remain yet to be discovered.

Notation and terminology

The standard unit vectors in the euclidean space $\mathbf{R}^n$, $n \geq 2$, are denoted by $e_1, \ldots, e_n$. A point x in $\mathbf{R}^n$ can be represented as a vector $(x_1, \ldots, x_n)$ or as a sum of vectors $x = x_1 e_1 + \ldots + x_n e_n$. For $x, y \in \mathbf{R}^n$ the inner product is defined by $x \cdot y = \sum_{i=1}^n x_i y_i$. The length (norm) of $x \in \mathbf{R}^n$ is $|x| = (x \cdot x)^{1/2}$. The ball centered at $x \in \mathbf{R}^n$ with radius $r > 0$ is $B^n(x, r) = \{ y \in \mathbf{R}^n : |x - y| < r \}$ and the sphere with the same center and radius is $S^{n-1}(x, r) = \{ y \in \mathbf{R}^n : |x - y| = r \}$. We employ the abbreviations

$$B^n(r) = B^n(0, r) \ , \quad \mathbf{B}^n = B^n(1) \ ,$$
$$S^{n-1}(r) = S^{n-1}(0, r) \ , \quad S^{n-1} = S^{n-1}(1) \ .$$

The n-dimensional volume of $\mathbf{B}^n$ is denoted by Ω_n and the $(n-1)$-dimensional surface area of S^{n-1} by ω_{n-1}. For $x, y \in \mathbf{R}^n$ let $[x, y] = \{ (1-t)x + ty : 0 \leq t \leq 1 \}$ and for $x \in \mathbf{R}^n \setminus \{0\}$ let $[x, \infty] = \{ sx : s \geq 1 \} \cup \{\infty\}$. The Möbius space $\overline{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$ is the one-point compactification of $\mathbf{R}^n$. The Möbius space, equipped with the spherical chordal distance q, is a metric space. In addition to $(\mathbf{R}^n, |\ |)$ and $(\overline{\mathbf{R}}^n, q)$ we shall require some other metric spaces such as the hyperbolic spaces $(\mathbf{B}^n, \rho_{\mathbf{B}^n})$ and $(\mathbf{H}^n, \rho_{\mathbf{H}^n})$ as well as (G, k_G) where $G \subset \mathbf{R}^n$ is a domain and k_G is the quasihyperbolic metric on G.

For a metric space (X, d) let $B_X(y, r) = \{ x \in X : d(x, y) < r \}$. If $A, B \subset X$ are non-empty let $d(A, B) = \inf\{ d(x, y) : x \in A, y \in B \}$ and $d(A) = \sup\{ d(x, y) : x, y \in A \}$. For $x \in X$ set $d(x, A) = d(\{x\}, A)$.

The set of natural numbers $0, 1, 2, \ldots$ is denoted by $\mathbf{N}$ and the set of all integers by $\mathbf{Z}$. The set of complex numbers is denoted by $\mathbf{C}$. We often identify $\mathbf{C} = \mathbf{R}^2$.

For a set A in $\mathbf{R}^n$ or $\overline{\mathbf{R}}^n$ the topological operations $\overline{A}$ (closure), ∂A (boundary), $\overline{\mathbf{R}}^n \setminus A$ (complement) are always taken with respect to $\overline{\mathbf{R}}^n$. Thus the domain $\mathbf{R}^n \setminus \{0\}$ has two boundary points, 0 and ∞, and the half-space $\mathbf{H}^n = \{ x \in \mathbf{R}^n : x_n > 0 \}$ has ∞ as a boundary point. A domain is an open connected non-empty set. A neighborhood of a point is a domain containing it. The notation $f \colon D \to D'$ usually includes the assumption that D and D' are domains in $\overline{\mathbf{R}}^n$.

Let G be an open set in $\mathbf{R}^n$. A mapping $f\colon G \to \mathbf{R}^m$ is differentiable at $x \in G$ if there exists a linear mapping $f'(x)\colon \mathbf{R}^n \to \mathbf{R}^m$, called the derivative of f at x, such that

$$f(x+h) = f(x) + f'(x)h + |h|\epsilon(x,h)$$

where $\epsilon(x,h) \to 0$ as $h \to 0$. The Jacobian determinant of f at x is denoted by $J_f(x)$. Assume next that $n = m$ and that all the partial derivatives exist at $x \in G$ (thus f need not be differentiable at x). In this case one defines the formal derivative of $f = (f_1, \ldots, f_n)$ at x as the linear map defined by

$$f'(x)e_i = \nabla f_i(x) = \left(\frac{\partial f_i}{\partial x_1}(x), \ldots, \frac{\partial f_i}{\partial x_n}(x)\right), \quad i = 1, \ldots, n .$$

For an open set $D \subset \mathbf{R}^n$ and for $k \in \mathrm{N}$, $C^k(D)$ denotes the set of all those continuous real-valued functions of D whose partial derivatives of order $p \le k$ exist and are continuous.

The n-dimensional volume of the unit ball $m_n(\mathbf{B}^n)$ is denoted by Ω_n and the $(n-1)$-dimensional surface area of S^{n-1} by ω_{n-1}. Then $\omega_{n-1} = n\Omega_n$ and

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma(1+\frac{1}{2}n)}$$

for all $n = 2, 3, \ldots$ where Γ stands for Euler's gamma function. For $k = 1, 2, \ldots$ we have by the well-known properties of the gamma function [AS, 6.1]

$$\omega_{2k-1} = \frac{2\pi^k}{(k-1)!}\,; \qquad \omega_{2k} = \frac{2^{k+1}\pi^k}{1 \cdot 3 \cdots (2k-1)} .$$

Algorithms suitable for numerical computation of $\Gamma(s)$ are given in [AS, Ch. 6] and in [PFTV, Ch. 6].

We next give a list of the additional notation used.

$\mathbf{H}^n = \mathbf{R}^n_+$	the Poincaré half-space *1*
$P(a,t)$	an $(n-1)$-dimensional hyperplane *2*
$\mathcal{GM}$	the group of Möbius transformations *3*
$\mathcal{O}(n)$	the group of orthogonal mappings *3*
$\mathcal{M}$	the group of sense-preserving Möbius transformations *3*
$\approx{x}$, $\approx{f}$	a generic point of $\{x \in \mathbf{R}^{n+1} : x_{n+1} = 0\}$ *4*
$\pi(x)$, $\pi_2(x)$	the stereographic projection *4, 6*

$q(x,y)$	the spherical (chordal) distance between x and y *4, 5*
$\widetilde{x}$	the antipodal (diametrically opposite) point *5*
$Q(x,r)$	the spherical ball *7*
$\lvert a,b,c,d \rvert$	the absolute (cross) ratio *9*
a^*	the image of a point a under an inversion in S^{n-1} *10*
T_a	a hyperbolic isometry with $T_a(a)=0$ *11*
$\mathrm{Lip}(f)$	the Lipschitz constant of f *11*
t_x	a spherical isometry with $t_x(x)=0$ *14*
$\rho(x,y)$	the hyperbolic distance between x and y *20, 23*
$J[x,y]$	the geodesic segment joining x and y in $\mathbf{R}^n_+$ *21*
$D(x,M)$	the hyperbolic ball with center x and radius M *22, 24*
$j_D(x,y)$	a point–pair function (metric) *28*
$k_D(x,y)$	the quasihyperbolic distance between x and y *33*
$D_G(x,M)$	the quasihyperbolic ball with center x and radius M *35*
$s_G(x,y)$	a point–pair function *39*
$p_X(A,t)$	the number of balls in a covering of the set A *46*
$\lvert\gamma\rvert$	the locus of a path *49*
$\ell(\gamma)$	the length of a curve γ *49*
$\mathsf{M}_p(\Gamma)$, $\mathsf{M}(\Gamma)$	the (p–)modulus of a curve family Γ *49*
$\Delta(E,F;G)$	the family of all closed non–constant curves joining E and F in G *51*
$\Delta(E,F)$	*52*
c_n	the constant in the spherical cap inequality *59*
$R_{G,n}(s)$	the Grötzsch ring *65*
$R_{T,n}(s)$	the Teichmüller ring *65*
$\gamma_n(s)=\gamma(s)$	the capacity of $R_{G,n}(s)$ *66*
$\tau_n(s)=\tau(s)$	the capacity of $R_{T,n}(s)$ *66*
$\mu(r)$	a function related to the complete elliptic integrals *67*
$\varphi_{K,n}(r)$	a special function related to the Schwarz lemma *68, 97*
$c(E)$	a set function related to the modulus *74*

p-cap E, cap E	the (p–)capacity of a condenser *82*
$\Lambda_\alpha(F)$	the α–dimensional Hausdorff measure of F *86*
$\Phi_n(s)$	the modulus of the Grötzsch ring *88*
$\Psi_n(s)$	the modulus of the Teichmüller ring *88*
λ_n	the Grötzsch ring constant *88*
$r_G(x,y)$	a point–pair function *102*
$\lambda_G(x,y)$	a conformal invariant (introduced by J. Ferrand) *103, 118*
$\mu_G(x,y)$	the modulus (conformal) metric *103*
$p(x)$	a function related to an extremal problem *106*
$m_G(x,y)$	a point–pair invariant *116*
$\mu(y,f,D)$, $\mu(f,D)$	the topological degree *121, 123*
B_f	the branch set of a mapping f *122*
$\dim E$	the topological dimension of a set E *123*
$J(G)$	the collection of all relatively compact subdomains of a domain G *123*
$i(x,f)$	the local (topological) index of f at x *123*
$U(x,f,r)$	a normal neighborhood of x *124*
$N(f,A)$	the maximal multiplicity of f in A *125*
$K(f)$, $K_O(f)$, $K_I(f)$	the maximal, outer, and inner dilatations of f *128*
$H(x,f)$	the linear dilatation of a mapping f at x *134*
$\lambda(K)$	a special function related to the linear dilatation *136*
$C(f,b)$	the cluster set of a mapping f at b *174*
$\mathrm{cap}\,\underline{\mathrm{dens}}(E,0)$	the lower capacity density of E at 0 *178*
$\mathrm{cap}\,\overline{\mathrm{dens}}(E,0)$	the upper capacity density of E at 0 *178*
$\mathrm{rad}\,\underline{\mathrm{dens}}(E,0)$	the lower radial density of E at 0 *178*
$\mathrm{rad}\,\overline{\mathrm{dens}}(E,0)$	the upper radial density of E at 0 *178*
$\mathrm{Dir}(u)$	the Dirichlet integral of u *187*

Chapter I
CONFORMAL GEOMETRY

This chapter is devoted to a study of some geometric quantities that remain invariant under the action of the group of Möbius transformations or under one of its subgroups. Examples of such subgroups are (1) translations, (2) orthogonal maps, (3) self-maps of $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n : x_n > 0\}$, and (4) spherical isometries. The Möbius invariance of the absolute (cross) ratio is of fundamental importance in such studies.

The following three metric spaces will be central to our discussions: (a) the euclidean space $\mathbf{R}^n$, (b) the Poincaré half-space $\mathbf{R}^n_+ = \mathbf{H}^n$, and (c) the Möbius space $\overline{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$. Each of these metric spaces is endowed with its own natural metric that is invariant under rigid motions of the space. In the particular case of $\mathbf{R}^n_+$, the invariant (hyperbolic) metric is often convenient in computations.

This chapter is partly expository in character. Some results, for instance various well-known properties of Möbius transformations in $\overline{\mathbf{R}}^n$, are presented without proofs. For these results and further information on Möbius transformations the reader is referred to Chapter 3 in A. F. Beardon's book [BE] as well as to L. V. Ahlfors' lecture notes [A5].

1. Möbius transformations in n–space

For $x \in \mathbf{R}^n$ and $r > 0$ let

$$B^n(x,r) = \{\, z \in \mathbf{R}^n : |x - z| < r\,\},$$

$$S^{n-1}(x,r) = \{\, z \in \mathbf{R}^n : |x - z| = r\,\}$$

denote the ball and sphere, respectively, centered at x with radius r. The abbreviations $B^n(r) = B^n(0,r)$, $S^{n-1}(r) = S^{n-1}(0,r)$, $\mathbf{B}^n = B^n(1)$, $S^{n-1} = S^{n-1}(1)$ will be used frequently. For $t \in \mathbf{R}$ and $a \in \mathbf{R}^n \setminus \{0\}$ we denote

$$P(a,t) = \{ x \in \mathbf{R}^n : x \cdot a = t \} \cup \{\infty\}.$$

Then $P(a,t)$ is a hyperplane in $\overline{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$ perpendicular to the vector a, at distance $t/|a|$ from the origin.

1.1. Definition. Let D and D' be domains in $\mathbf{R}^n$ and let $f\colon D \to D'$ be a homeomorphism. We call f *conformal* if (1) $f \in C^1$, (2) $J_f(x) \neq 0$ for all $x \in D$, and (3) $|f'(x)h| = |f'(x)|\,|h|$ for all $x \in D$ and all $h \in \mathbf{R}^n$. If D and D' are domains in $\overline{\mathbf{R}}^n$, we call a homeomorphism $f\colon D \to D'$ *conformal* if the restriction of f to $D \setminus \{\infty, f^{-1}(\infty)\}$ is conformal.

1.2. Examples. Some basic examples of conformal mappings are the following elementary transformations.

(1) A reflection in $P(a,t)$:

$$f_1(x) = x - 2(x \cdot a - t)\frac{a}{|a|^2}\,,\ f_1(\infty) = \infty\ .$$

(2) An inversion (reflection) in $S^{n-1}(a,r)$:

$$f_2(x) = a + \frac{r^2(x-a)}{|x-a|^2}\,,\ f_2(a) = \infty\,,\ f_2(\infty) = a\ .$$

(3) A translation $f_3(x) = x + a$, $a \in \mathbf{R}^n$, $f_3(\infty) = \infty$.

(4) A stretching by a factor $k > 0$: $f_4(x) = kx$, $f_4(\infty) = \infty$.

(5) An orthogonal mapping, i.e. a linear map f_5 with

$$|f_5(x)| = |x|\,,\ f_5(\infty) = \infty\ .$$

1.3. Remark. The translation $x \mapsto x + a$ can be written as a composition of reflections in $P(a,0)$ and $P(a, \frac{1}{2}|a|^2)$. The stretching $x \mapsto kx$, $k > 0$, can be written as a composition of inversions in $S^{n-1}(0,1)$ and $S^{n-1}(0,\sqrt{k})$. It can be proved, that an orthogonal mapping can be composed of at most $n+1$ reflections in planes (see [BE, p. 23, Theorem 3.1.3]).

1.4. Exercise. Let f be an inversion in $S^{n-1}(a,r)$ as defined in 1.2(2). Show that $f^{-1} = f$ and that $|x-a||f(x)-a| = r^2$ for all $x \in \mathbf{R}^n \setminus \{a\}$. By considering similar triangles show that the following identity holds for $x, y \in \mathbf{R}^n \setminus \{a\}$:

$$|f(x) - f(y)| = \frac{r^2|x-y|}{|x-a||y-a|}\ . \tag{1.5}$$

1.6. Exercise. For $x, y \in \mathbf{R}^n \setminus \{0\}$ let $p(x,y) = |x-y|^2/(|x||y|)$. Applying (1.5) show that $p(x,y) = p(f(x), f(y))$ if f is a stretching or an inversion in $S^{n-1}(r)$, $r > 0$.

1.7. Definition. A homeomorphism $f\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ is called a *Möbius transformation* if $f = g_1 \circ \cdots \circ g_p$ where each g_j is one of the elementary transformations in 1.2(1)–(5) and p is a positive integer. Equivalently (see 1.3) f is a Möbius transformation if $f = h_1 \circ \cdots \circ h_m$ where each h_j is a reflection in a sphere or in a hyperplane and m is a positive integer.

It follows from the inverse function theorem and the chain rule that the set of all conformal mappings of $\overline{\mathbf{R}}^n$ is a group. It is left as an easy exercise for the reader to show that the set of all Möbius transformations constitutes a subgroup of the group of conformal mappings, and we denote it by $\mathcal{GM}(\overline{\mathbf{R}}^n)$ or $\mathcal{GM}$. Further, we shall write

$$\mathcal{GM}(D) = \{ f \in \mathcal{GM}(\overline{\mathbf{R}}^n) : fD = D \}$$

for $D \subset \overline{\mathbf{R}}^n$. We denote by $\mathcal{O}(n)$ the set of all orthogonal maps in $\mathbf{R}^n$. A map f in $\mathcal{GM}$ with $f(\infty) = \infty$ is called a *similarity* transformation if $|f(x) - f(y)| = c|x-y|$ for all $x, y \in \mathbf{R}^n$ where c is a positive number.

1.8. Definition. Let D and D' be domains in $\overline{\mathbf{R}}^n$. We call a C^1-homeomorphism $f\colon D \to D'$ *sense-preserving* (orientation-preserving) if $J_f(x) > 0$ for all $x \in D \setminus \{\infty, f^{-1}(\infty)\}$. If $J_f(x) < 0$ for all $x \in D \setminus \{\infty, f^{-1}(\infty)\}$ then we call f *sense-reversing* (orientation-reversing).

One can show that reflection in a hyperplane or in a sphere is sense-reversing and hence the composition of an odd number of reflections. The composition of an even number of reflections is sense-preserving. For these results the reader is referred to [RR, pp. 137–145]. The set of all sense-preserving Möbius transformations is denoted by $\mathcal{M}(\overline{\mathbf{R}}^n)$ or $\mathcal{M}$. Also we let $\mathcal{M}(D) = \{ f \in \mathcal{M} : fD = D \}$ if $D \subset \overline{\mathbf{R}}^n$.

1.9. Remark. One can extend Definition 1.8 so as to make it applicable to a wider class of mappings (including quasiregular mappings). This extended definition makes use of the topological degree of a mapping, which will be briefly discussed in Section 9.

It will be convenient to identify $\overline{\mathbf{R}}^n$ with the subset $\{ x \in \mathbf{R}^n : x_{n+1} = 0 \} \cup \{\infty\}$ of $\overline{\mathbf{R}}^{n+1}$. The identification is given by the embedding

$$x \mapsto \tilde{\tilde{x}} = (x_1, \ldots, x_n, 0) \; ; \quad x = (x_1, \ldots, x_n) \in \mathbf{R}^n \, . \tag{1.10}$$

We are now going to describe a natural two–step way of extending a Möbius transformation of $\overline{\mathbf{R}}^n$ to a Möbius transformation of $\overline{\mathbf{R}}^{n+1}$. First, if f in $\mathcal{GM}(\overline{\mathbf{R}}^n)$ is a reflection in $P(a,t)$ or in $S^{n-1}(a,r)$, let $\tilde{\tilde{f}}$ be a reflection in $P(\tilde{\tilde{a}},t)$ or $S^n(\tilde{\tilde{a}},r)$, respectively. Then if $x \in \overline{\mathbf{R}}^n$ and $y = f(x)$, by 1.2(1)–(2) we get

$$\tilde{\tilde{f}}(x_1, \ldots, x_n, 0) = (y_1, \ldots, y_n, 0) = \widetilde{\widetilde{f(x)}} \, . \tag{1.11}$$

By (1.11) we may regard $\tilde{\tilde{f}}$ as an extension of f. Note that $\tilde{\tilde{f}}$ preserves the plane $x_{n+1} = 0$ and each of the half spaces $x_{n+1} > 0$ and $x_{n+1} < 0$. These facts follow from the formulae 1.2(1)–(2). Second, if f is an arbitrary mapping in $\mathcal{GM}(\overline{\mathbf{R}}^n)$ it has a representation $f = f_1 \circ \cdots \circ f_m$ where each f_j is a reflection in a plane or a sphere. Then $\tilde{\tilde{f}} = \tilde{\tilde{f}}_1 \circ \cdots \circ \tilde{\tilde{f}}_m$ is the extension of f, and it preserves the half spaces $x_{n+1} > 0$, $x_{n+1} < 0$, and the plane $x_{n+1} = 0$. In conclusion, every f in $\mathcal{GM}(\overline{\mathbf{R}}^n)$ has an extension $\tilde{\tilde{f}}$ in $\mathcal{GM}(\overline{\mathbf{R}}^{n+1})$. It follows from [BE, p. 31, Theorem 3.2.4] that such an extension $\tilde{\tilde{f}}$ of f is unique. The mapping $\tilde{\tilde{f}}$ is called the *Poincaré extension* of f. In the sequel we shall write x, f instead of $\tilde{\tilde{x}}$, $\tilde{\tilde{f}}$, respectively.

Many properties of plane Möbius transformations hold for n–dimensional Möbius transformations as well. The fundamental property that spheres of $\overline{\mathbf{R}}^n$ (which are spheres or planes in $\mathbf{R}^n$, see Exercise 1.25 below) are preserved under Möbius transformations is proved in [BE, p. 28, Theorem 3.2.1].

1.12. Stereographic projection. The *stereographic projection* $\pi \colon \overline{\mathbf{R}}^n \to S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ is defined by

$$\pi(x) = e_{n+1} + \frac{x - e_{n+1}}{|x - e_{n+1}|^2} \, , \; x \in \mathbf{R}^n \; ; \; \pi(\infty) = e_{n+1} \, . \tag{1.13}$$

Then π is the restriction to $\overline{\mathbf{R}}^n$ of the inversion in $S^n(e_{n+1}, 1)$. In fact, we can identify π with this inversion. Because $f^{-1} = f$ for every inversion f, it follows that π maps the "Riemann sphere" $S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ onto $\overline{\mathbf{R}}^n$.

The *spherical (chordal) metric* q in $\overline{\mathbf{R}}^n$ is defined by

$$q(x,y) = |\pi(x) - \pi(y)| \; ; \; x, \, y \in \overline{\mathbf{R}}^n \, , \tag{1.14}$$

where π is the stereographic projection (1.13). From the definition (1.13) and by (1.5) we obtain

$$(1.15)\qquad \begin{cases} q(x,y) = \dfrac{|x-y|}{\sqrt{1+|x|^2}\,\sqrt{1+|y|^2}}\ ; \ x \neq \infty \neq y\ , \\ q(x,\infty) = \dfrac{1}{\sqrt{1+|x|^2}}\ . \end{cases}$$

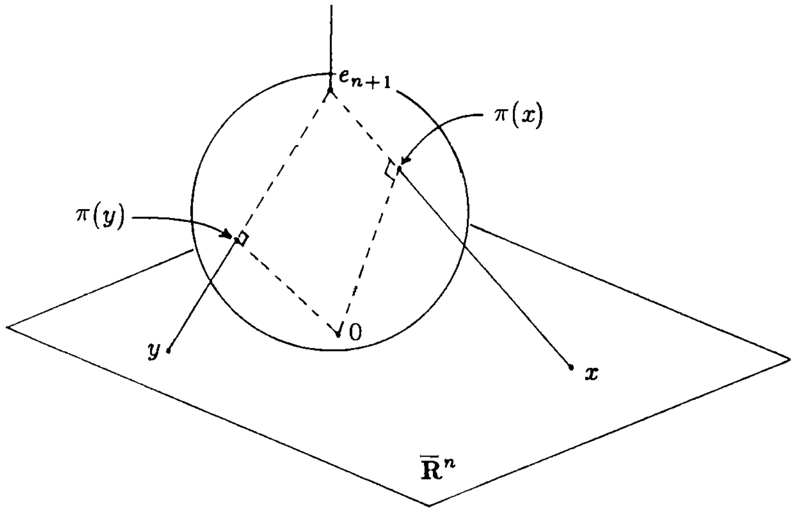


Diagram 1.1. Formulae (1.13) and (1.14) visualized.

For $x \in \mathbf{R}^n \setminus \{0\}$ the *antipodal* (diametrically opposite) *point* $\widetilde{x}$ is defined by

$$(1.16)\qquad \widetilde{x} = -\frac{x}{|x|^2}$$

and we set $\widetilde{\infty} = 0$, $\widetilde{0} = \infty$. Then, by (1.15), $q(x,\widetilde{x}) = 1$ and hence $\pi(x)\,,\pi(\widetilde{x})$ are indeed diametrically opposite points on the Riemann sphere.

1.17. Exercise. It follows from (1.15) that $q(x,y) \le \min\{1, |x-y|\}$ is always true. Applying (1.5) show that

$$q(x,y) = q\Big(\frac{x}{|x|^2}, \frac{y}{|y|^2}\Big) \le \frac{|x-y|}{|x||y|}$$

holds for $x, y \in \mathbf{R}^n \setminus \{0\}$. Show also that

$$\frac{|x-y|}{(1+|x|)(1+|y|)} \le q(x,y) \le \frac{2|x-y|}{(1+|x|)(1+|y|)}$$

for all $x, y \in \mathbf{R}^n$. Show that $q(x,y) \ge \frac{1}{2}|x-y|$ for $x, y \in \overline{\mathbf{B}}^n$ and that $q(x,y) = \frac{1}{2}|x-y|$ for $x, y \in \partial\mathbf{B}^n$.

1.18. Exercise. (1) For $0 < t < 1$ let $w(t) = t/\sqrt{1-t^2}$. Show that $q(0, w(t)e_1) = t$ and that

$$\frac{t}{s} < \frac{w(t)}{w(s)} < \frac{2t}{s}$$

for $0 < s < t < \frac{1}{2}\sqrt{3}$.

(2) Let $x, y \in \mathbf{B}^n$ with $s = q(0,x)$, $t = q(0,y)$. Show that

$$q(x,y) \le s\sqrt{1-t^2} + t\sqrt{1-s^2} \le t + s .$$

(3) Let $x, y \in \mathbf{R}^n \setminus \{0\}$ with $q(0,x) > q(0,y)$. Show that the strict inequality $q(x,y) > q(0,x) - q(0,y)$ holds.

(4) Show that for $x, y, z \in \mathbf{B}^n$, $x \ne z$,

$$\frac{1}{\sqrt{2}}\frac{|x-y|}{|x-z|} \le \frac{q(x,y)}{q(x,z)} \le \sqrt{2}\frac{|x-y|}{|x-z|} .$$

1.19. Definition. Let (X_1, d_1) and (X_2, d_2) be metric spaces and let $f\colon X_1 \to X_2$ be a homeomorphism. We call f an *isometry* if $d_2(f(x), f(y)) = d_1(x,y)$ for all $x, y \in X_1$. A map f in $\mathcal{GM}(\overline{\mathbf{R}}^n)$ is called a *spherical isometry* if $q(f(x), f(y)) = q(x,y)$ for all $x, y \in \overline{\mathbf{R}}^n$. A similarity transformation f is called a *euclidean isometry* if $|f(x) - f(y)| = |x - y|$ for all $x, y \in \mathbf{R}^n$.

Orthogonal mappings and inversion in S^{n-1} are examples of spherical isometries, while the translation $x \mapsto x + e_1$ and the stretching $x \mapsto \frac{1}{2}x$ are not spherical isometries (see 1.54). Reflection in a hyperplane and translations are examples of euclidean isometries.

1.20. Remark. The inversion

$$\pi_2(x) = e_{n+1} + \frac{2(x - e_{n+1})}{|x - e_{n+1}|^2} , \; x \in \mathbf{R}^n ; \; \pi_2(\infty) = e_{n+1} ,$$

is also sometimes called the stereographic projection. It maps $\overline{\mathbf{R}}^n$ onto S^n so that $\pi_2(0) = -e_{n+1}$, $\pi_2(\infty) = e_{n+1}$ and $\pi_2(S^{n-1}) = S^{n-1}$. From (1.5) it follows that the spherical metric q in (1.14) can be defined in terms of π_2 as $q(x,y) = \frac{1}{2}|\pi_2(x) - \pi_2(y)|$, $x, y \in \overline{\mathbf{R}}^n$. We can identify π_2 with the inversion in $S^n(e_{n+1}, \sqrt{2})$. We see that π_2 maps the half space $x_{n+1} < 0$ onto $\mathbf{B}^{n+1}$ in such a way that $\pi_2(-e_{n+1}) = 0$, $\pi_2(0) = -e_{n+1}$.

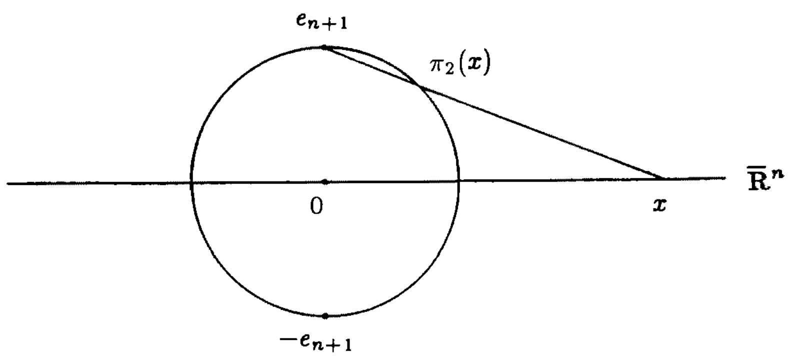


Diagram 1.2.

1.21. Balls in the spherical metric. For $x \in \overline{\mathbf{R}}^n$ and $r \in (0,1)$ we define the spherical ball

$$Q(x,r) = \{ z \in \overline{\mathbf{R}}^n : q(x,z) < r \} . \tag{1.22}$$

Its boundary sphere is denoted by $\partial Q(x,r)$. From the Pythagorean theorem it follows that (cf. (1.16))

$$Q(x,r) = \overline{\mathbf{R}}^n \setminus \overline{Q}(\tilde{x}, \sqrt{1-r^2}) . \tag{1.23}$$

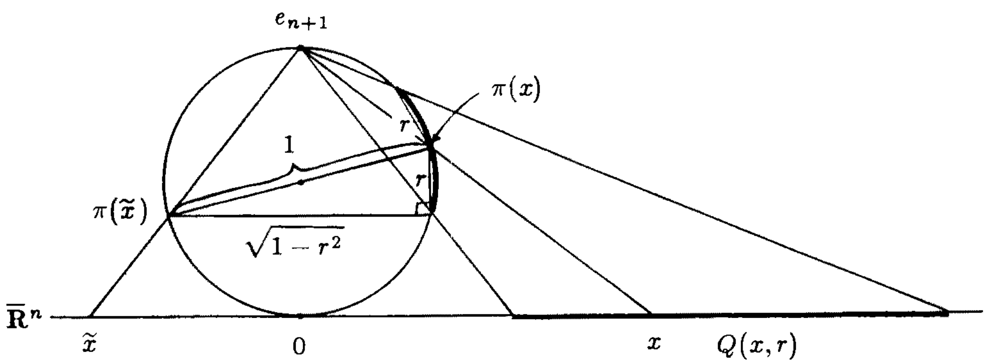


Diagram 1.3.

To gain insight into the geometry of spherical balls $Q(x,r)$ it is convenient to study the image $\pi Q(x,r)$ under the stereographic projection π (see the diagram). Indeed, by definition (1.14) we see that

$$\pi Q(x,r) = B^{n+1}(\pi(x),r) \cap S^n(\tfrac{1}{2}e_{n+1}, \tfrac{1}{2}).$$

Either by this formula or more directly by the definition of the spherical metric (plus the fact that Möbius transformations preserve spheres) we see that in the euclidean geometry, $Q(x,r)$ is a point set of one of the following three kinds

(a) an open ball $B^n(u,s)$,

(b) the complement of $\overline{B}^n(v,t)$ in $\overline{\mathbf{R}}^n$,

(c) a half-space of $\mathbf{R}^n$.

Clearly, $\partial Q(x,r)$ is either a sphere or a hyperplane of $\mathbf{R}^n$. Formula (1.23) shows, in particular, that $\pi Q(x,1/\sqrt{2})$ is a half-sphere of the Riemann sphere $S^n(\frac{1}{2}e_{n+1},\frac{1}{2})$.

1.24. Remark. In complex analysis, the spherical metric is often defined in the following way. If γ is a rectifiable curve in $\mathbf{R}^n$ set

$$\sigma(\gamma) = \int_\gamma \frac{|dx|}{1+|x|^2}$$

and for $x,y \in \mathbf{R}^n$ define

$$\sigma(x,y) = \inf_\gamma \sigma(\gamma)$$

where γ runs through the collection of all rectifiable curves γ with $x \in \gamma$ and $y \in \gamma$. In a natural way this definition is extended then to all $x,y \in \overline{\mathbf{R}}^n$. It is easy to show that $\sigma(x,y)$ is a metric on $\overline{\mathbf{R}}^n$. Making use of (1.15) one can show that $\sigma(x,y) = \sigma(h(x),h(y))$ if h is a spherical isometry. This spherical metric is equivalent to the metric q. In fact, the two relationships

$$\sigma(x,y) = 2\arcsin q(x,y) \,,$$
$$1 \le \frac{\sigma(x,y)}{q(x,y)} \le \pi = 4\arctan 1 \,,$$

hold for all distinct $x,y \in \overline{\mathbf{R}}^n$.

1.25. Exercise. (1) Show that $\{ x \in \mathbf{R}^n : x_n > 0 \} = Q(e_n, 1/\sqrt{2})$, $\mathbf{B}^n = Q(0, 1/\sqrt{2})$, $\overline{\mathbf{R}}^n \setminus \mathbf{B}^n = Q(\infty, 1/\sqrt{2})$, $Q(0,r) = B^n(r/\sqrt{1-r^2})$, and $B^n(t) = Q(0, t/\sqrt{1+t^2})$. If $t \in (0,1)$, $x = te_1$, $y = -\frac{1-t}{1+t}e_1$, show that $x, \tilde{x}, e_2, -e_2 \in \partial Q(y, 1/\sqrt{2})$.

(2) Show that $Q(te_1, 1/\sqrt{2}) = B^n(ue_1, r)$, where $u = 2t/(1-t^2)$, $r = (1+t^2)/(1-t^2)$, provided $t \in (0,1)$. Discuss also the case $t \in (1,\infty)$.

(3) Find $q(Q(x,r))$ and $q(\partial Q(x,r))$, the spherical diameters of $Q(x,r)$ and $\partial Q(x,r)$, respectively, for $x \in \overline{\mathbf{R}}^n$, $r \in (0,1)$. [Hint: Consider first the case $r < 1/\sqrt{2}$.]

(4) Show that if $z \in \mathbf{R}^n$, then there exists $e \in S^{n-1}$ such that e and $-e$ are in $\partial Q(z, 1/\sqrt{2})$. Conclusion: If $Q(z, 1/\sqrt{2}) = B^n(a,r)$, then $r^2 = 1 + |a|^2$. Show conversely that $q\big(B^n(b, \sqrt{1+|b|^2})\big) = 1$ for all $b \in \mathbf{R}^n$.

1.26. Absolute ratio. For an ordered quadruple a,b,c,d of distinct points in $\overline{\mathbf{R}}^n$ we define the *absolute (cross) ratio* by

$$|\,a,b,c,d\,| = \frac{q(a,c)\,q(b,d)}{q(a,b)\,q(c,d)}\,. \tag{1.27}$$

It follows from (1.15) that for distinct a,b,c,d in $\mathbf{R}^n$

$$|\,a,b,c,d\,| = \frac{|a-c|\,|b-d|}{|a-b|\,|c-d|}\,.$$

One of the most important properties of Möbius transformations is that they *preserve absolute ratios*, i.e. if $f \in \mathcal{GM}$, then

$$|\,f(a), f(b), f(c), f(d)\,| = |\,a,b,c,d\,| \tag{1.28}$$

for all distinct a,b,c,d in $\overline{\mathbf{R}}^n$. As a matter of fact, the preservation of absolute ratios is a characteristic property of Möbius transformations. It is proved in [BE, p. 72, Theorem 3.2.7] that a mapping $f\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ is a Möbius transformation if and only if f preserves all absolute ratios. It follows from (1.28) that $|\,a,b,c,d\,| = \lambda$ if and only if there exists an f in $\mathcal{GM}$ with

$$f(a) = 0\,,\ f(b) = e_1\,,\ f(d) = \infty\,,\ |f(c)| = \lambda\,. \tag{1.29}$$

By property (1.28), the absolute ratio is $\mathcal{GM}(\overline{\mathbf{R}}^n)$-invariant. Besides the absolute ratio we shall consider later some other $\mathcal{GM}(\overline{\mathbf{R}}^n)$- or $\mathcal{GM}(D)$-invariant quantities. For instance, $|a-b|/|a-c|$ and $d(E)/d(0,E)$, $E \subset \mathbf{R}^n \setminus \{0\}$, are $\mathcal{GM}(\mathbf{R}^n)$-invariant quantities.

1.30. Remark. The absolute ratio depends on the order of the points. In fact,

$$\begin{aligned}
|\,0, e_1, x, \infty\,| &= |x| = \frac{1}{|\,0, x, e_1, \infty\,|}\,,\\
|\,0, e_1, \infty, x\,| &= |x - e_1| = \frac{1}{|\,0, \infty, e_1, x\,|}\,,\\
|\,0, \infty, x, e_1\,| &= \frac{|x|}{|x-e_1|} = \frac{1}{|\,0, x, \infty, e_1\,|}\,.
\end{aligned}$$

A thorough discussion of the complex cross–ratio can be found in [BE, pp. 75–78].

1.31. Exercise. Let f be in $\mathcal{GM}$ such that $f(0)=0$, $f(e_1)=e_1$, and $f(\infty)=\infty$. Show that f is an orthogonal mapping. [Hint: Apply (1.28).]

1.32. Exercise. For $a,b,c,d \in \overline{\mathbf{R}}^n$ let

$$s(a,b,c,d) = \frac{q(a,d)^2\, q(b,c)^2}{q(a,b)\, q(b,d)\, q(a,c)\, q(c,d)}\,,$$
$$s_f(a,b,c,d) = s\big(f(a),f(b),f(c),f(d)\big)\,.$$

Then s is symmetric: $s(a,b,c,d)=s(d,b,c,a)=s(b,a,d,c)=s(a,c,b,d)$. Show that s is $\mathcal{GM}(\overline{\mathbf{R}}^n)$-invariant, i.e.

$$s_f(a,b,c,d) = s(a,b,c,d) \tag{1.33}$$

whenever $a,b,c,d \in \overline{\mathbf{R}}^n$ and $f \in \mathcal{GM}$. Applying (1.15) show that $s(0,x,y,\infty) = |x-y|^2/(|x||y|)$. It should be noted that the invariance property in Exercise 1.6 is a special case of (1.33). [Hint: Show that $s(a,b,c,d)$ is the product of two absolute ratios.]

1.34. Automorphisms of $\mathbf{B}^n$. We shall give a canonical representation for the maps in $\mathcal{M}(\mathbf{B}^n)$. Assume that f is in $\mathcal{M}(\mathbf{B}^n)$ and that $f(a)=0$ for some $a\in\mathbf{B}^n$. We denote

$$a^* = \frac{a}{|a|^2}\,,\quad a\in\mathbf{R}^n\setminus\{0\} \tag{1.35}$$

and $0^*=\infty$, $\infty^*=0$. Fix $a\in\mathbf{B}^n\setminus\{0\}$. Let

$$\sigma_a(x) = a^* + r^2(x-a^*)^*\,,\; r^2 = |a|^{-2}-1 \tag{1.36}$$

be an inversion in the sphere $S^{n-1}(a^*,r)$ orthogonal to S^{n-1}. Then $\sigma_a(a)=0$, $\sigma_a(a^*)=\infty$.

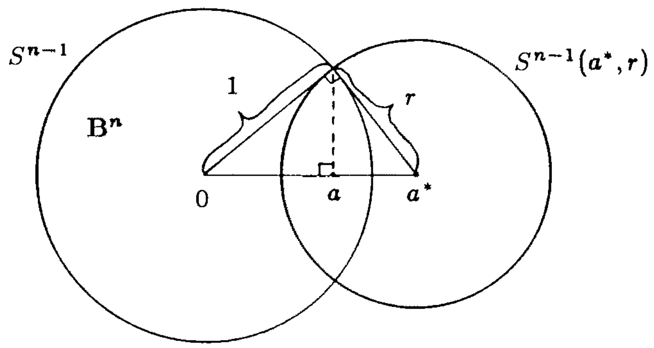


Diagram 1.4.

Let p_a denote the reflection in the $(n-1)$-dimensional plane $P(a,0)$ through the origin and orthogonal to a and define a sense-preserving Möbius transformation by $T_a = p_a \circ \sigma_a$. Then, by (1.36), $T_a \mathbf{B}^n = \mathbf{B}^n$, $T_a(a) = 0$, and with $e_a = a/|a|$ we have $T_a(e_a) = e_a$, $T_a(-e_a) = -e_a$. For $a = 0$ we set $T_0 = \mathrm{id}$, where id stands for the identity map. The proof of the following fundamental fact can be found in [A5, p. 21], [BE, p. 40, Theorem 3.5.1].

1.37. Lemma. *If* $g \in \mathcal{GM}(\mathbf{B}^n)$, *then there is* $k \in \mathcal{O}(n)$ *such that* $g = k \circ T_a$ *where* $a = g^{-1}(0)$.

1.38. Definition. Let (X_1, d_1), (X_2, d_2) be metric spaces, let $f\colon X_1 \to X_2$ be continuous and let $L \geq 1$. We say that f is *L-Lipschitz* if

$$d_2\big(f(x), f(y)\big) \leq L\, d_1(x,y)$$

for all $x, y \in X_1$. The least constant L with this property is denoted by $\mathrm{Lip}(f)$. If, in addition, f is a homeomorphism and

$$d_1(x,y)/L \leq d_2\big(f(x), f(y)\big) \leq L\, d_1(x,y)$$

for all $x, y \in X_1$, we say that f is *L-bilipschitz* or that f is an *L-quasiisometry.* We call f a *Lipschitz (bilipschitz)* mapping if it is L-Lipschitz (resp. L-bilipschitz) for some $L \geq 1$.

If $h \in \mathcal{GM}$ and $x \in \overline{\mathbf{R}}^n$ we sometimes write hx instead of $h(x)$.

1.39. The Lipschitz constant of $T_a|\mathbf{B}^n$. Let $T_a = p_a \circ \sigma_a$ be as in 1.34, $a \in \mathbf{B}^n \setminus \{0\}$. Since p_a is a reflection in a plane and hence preserves euclidean distances, it follows that $|T_a x - T_a y| = |\sigma_a x - \sigma_a y|$. As $|a|^{-1} - 1 \leq |z - a^*| \leq |a|^{-1} + 1$ for all $z \in \overline{\mathbf{B}}^n$, using (1.5) we get

$$|T_a x - T_a y| \leq \Big(\frac{|a|}{1-|a|}\Big)^2 \big(|a|^{-2} - 1\big)|x-y| = \frac{1+|a|}{1-|a|}\,|x-y|\ ,$$

$$|T_a x - T_a y| \geq \Big(\frac{|a|}{1+|a|}\Big)^2 \big(|a|^{-2} - 1\big)|x-y| = \frac{1-|a|}{1+|a|}\,|x-y|$$

for all $x, y \in \mathbf{B}^n$. Hence

$$\mathrm{Lip}(T_a|\mathbf{B}^n) = \sup\Big\{\frac{|T_a x - T_a y|}{|x-y|} \,:\, x, y \in \mathbf{B}^n,\, x \neq y\Big\} \leq \frac{1+|a|}{1-|a|}\ .$$

In fact,

$$\mathrm{Lip}(T_a|\mathbf{B}^n) = \mathrm{Lip}(T_a|S^{n-1}) = \frac{1+|a|}{1-|a|} \tag{1.40}$$

as we see by applying (1.5) to a pair of points $x, y \in S^{n-1}$ with $|x - a^*| = |y - a^*|$ and then letting $|x - a^*| \to |a|^{-1} - 1$. As $T_a^{-1} = T_{-a}$, it follows from (1.40) that $T_a|\mathbf{B}^n$ is bilipschitz with the constant $(1+|a|)/(1-|a|)$.

1.41. Exercise. (1) Show that

$$|T_x y| = \frac{|x-y|}{|x||y-x^*|}$$

for all $x, y \in \mathbf{B}^n$. [Hint: Apply (1.5).]

(2) Let $r \in (0,1)$. Find the "Möbius center" of the segment $[0, re_1]$, i.e. the point $a \in (0, re_1)$ such that $T_a(0) = -T_a(re_1)$ where T_a is as in 1.39. [Hint: First observe that $|T_a(a) - T_a(0)| = |T_a(re_1) - T_a(a)|$ and hence, by the definition of T_a, a similar equality holds with σ_a in place of T_a. Next apply (1.5) to σ_a to obtain $|a| = r/(1+\sqrt{1-r^2})$. Note that the point a can be found by a geometric construction, as in the next diagram.]

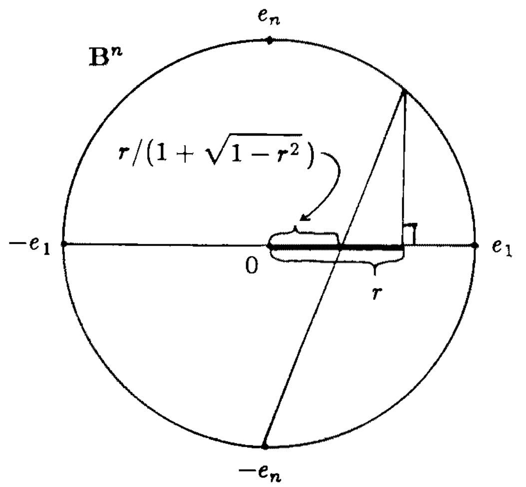


Diagram 1.5.

1.42. Exercise. (1) Show that if $\varphi \in (0, \frac{1}{2}\pi)$, $x_\varphi = (\cos\varphi, \sin\varphi)$, $y_\varphi = (\cos\varphi, -\sin\varphi)$ then there exists a Möbius transformation $T_a\colon \mathbf{B}^2 \to \mathbf{B}^2$ with $T_a e_1 =$

e_1 , $T_a(-e_1) = -e_1$, $T_a(x_\varphi) = e_2 = -T_a(y_\varphi)$, and $\mathrm{Lip}(T_a|\overline{\mathbf{B}}^2) = \cot \frac{1}{2}\varphi$. [Hint: By 1.39 we see that $a^* = \frac{1}{|a|}e_1$ and e_2, x_φ, a must be collinear. Now $\frac{1}{|a|} = \tan(\frac{1}{4}\pi + \frac{1}{2}\varphi)$, $\frac{1+|a|}{1-|a|} = \cot \frac{1}{2}\varphi$, and the result follows from (1.40).]

(2) Let $\varphi \in (0, \frac{1}{2}\pi)$, $F_\varphi = \{ x \in S^{n-1} : x_1 = \cos\varphi \}$, and let $T_a \in \mathcal{M}(\mathbf{B}^n)$ with $T_a F_\varphi = F_\varphi$. Show that $a = (\cot \frac{1}{2}\varphi)^2 e_1$. Assume next that $0 < \alpha \leq \beta < \frac{1}{2}\pi$ and $T_b F_\alpha = F_\beta$. Find b .

1.43. Exercise. (1) Let $0 < s < 1$. Applying (1.5) as in 1.39 show that

$$\frac{1-s^2}{(1+s^2)^2}|x-y| \leq |T_a x - T_a y| \leq \frac{1}{1-s^2}|x-y|$$

for all $a, x, y \in \overline{B}^n(s)$.

(2) For $a, x \in \mathbf{B}^n$ with $a \neq 0$ and $e_a = a/|a|$ show that

$$|T_a x| \leq |T_a(-|x|e_a)| \, .$$

1.44. Spherical isometries. We are now going to study the action of spherical isometries, proving a representation for them similar to that of $\mathcal{GM}(\mathbf{B}^n)$ in 1.37. By (1.40) and 1.37 we see that $g \in \mathcal{GM}(\mathbf{B}^{n+1})$ is a euclidean isometry iff $g(0) = 0$. Next we we shall reformulate this fact for maps in $\mathcal{GM}(\overline{\mathbf{R}}^n)$. Let p be the reflection in the hyperplane $x_{n+1} = 0$ and f_1 the inversion in $S^n(e_{n+1}, \sqrt{2})$, and set $f = f_1 \circ p$. Then $f\mathbf{R}^{n+1}_+ = \mathbf{B}^{n+1}$ and $f(e_{n+1}) = 0$, $q(x,y) = \frac{1}{2}|f(x) - f(y)|$ for all $x, y \in \overline{\mathbf{R}}^n$. Assume now that $h \in \mathcal{GM}(\overline{\mathbf{R}}^n)$ is given and that $\tilde{\tilde{h}} \in \mathcal{GM}(\overline{\mathbf{R}}^{n+1})$ is its Poincaré extension. We see that $\varphi = f \circ \tilde{\tilde{h}} \circ f^{-1} \in \mathcal{GM}(\mathbf{B}^{n+1})$ is a euclidean isometry if and only if $\tilde{\tilde{h}}(e_{n+1}) = e_{n+1}$. One can show that h is a spherical isometry iff $\tilde{\tilde{h}}(e_{n+1}) = e_{n+1}$ (see [BE, p. 42, Theorem 3.6.1]). In particular, the inversion in $S^{n-1}(a, r) \subset \mathbf{R}^n$ is a spherical isometry iff $e_{n+1} \in S^n(\tilde{\tilde{a}}, r) \subset \mathbf{R}^{n+1}$, i.e. iff

$$r^2 = 1 + |a|^2. \tag{1.45}$$

We recall (see (1.23), 1.25) that by virtue of (1.45) $B^n(a,r) = Q(z, 1/\sqrt{2})$ for some $z \in \mathbf{R}^n$.

We define a spherical isometry t_z in $\mathcal{M}(\overline{\mathbf{R}}^n)$ which maps a given point $z \in \overline{\mathbf{R}}^n$ to 0 as follows. For $z = 0$ let $t_z = \mathrm{id}$ and for $z = \infty$ let $t_z = p \circ f$, where f is inversion in S^{n-1} and p is reflection in the $(n-1)$-dimensional plane $x_1 = 0$. For $z \in \mathbf{R}^n \setminus \{0\}$ let s_z be inversion in $S^{n-1}(-z/|z|^2, r)$, where $r = \sqrt{1 + |z|^{-2}}$.

According to the criterion (1.45) the inversion s_z is a spherical isometry and it is easy to show that $s_z(z) = 0$. Let p_z be reflection in the plane $P(z, 0)$. Defining

$$t_z = p_z \circ s_z \tag{1.46}$$

we see that $t_z \in \mathcal{M}(\overline{\mathbf{R}}^n)$ is a spherical isometry with $t_z(z) = 0$. Hence

$$\begin{aligned} t_z(Q(z,r)) &= Q(0,r) = B^n\left(r/\sqrt{1-r^2}\right), \\ |t_z(x)|^2 &= \frac{q(x,z)^2}{1-q(x,z)^2} \end{aligned} \tag{1.47}$$

for all $x, z \in \overline{\mathbf{R}}^n$, $r \in (0,1)$.

In the above discussion we showed that the inversion s_z is a spherical isometry by exploiting a result from [BE]. Next we shall show this by a direct computation.

1.48. Lemma. *For a point $z \in \mathbf{R}^n \setminus \{0\}$ let s_z be the inversion in the sphere $S^{n-1}(-z/|z|^2, \sqrt{1+|z|^{-2}})$. Then $s_z(z) = 0$ and s_z is a spherical isometry.*

Proof. It is easy to show that $s_z(z) = 0$. By (1.5) we obtain for $x, y \in \mathbf{R}^n$

$$|s_z(x) - s_z(y)| = \frac{(1+|z|^{-2})|x-y|}{|x+z/|z|^2|\,|y+z/|z|^2|} = \frac{(1+|z|^{-2})|x-y|}{|x-\tilde{z}||y-\tilde{z}|},$$

where $\tilde{z} = -z/|z|^2$ as in (1.16). Further by (1.5)

$$\begin{aligned} |s_z(x)| &= |s_z(x) - s_z(z)| = \frac{(1+|z|^{-2})|x-z|}{|x-\tilde{z}||z-\tilde{z}|} = \frac{|x-z|}{|z||x-\tilde{z}|}, \\ |s_z(y)| &= \frac{|y-z|}{|z||y-\tilde{z}|}. \end{aligned}$$

Substituting these identities into (1.15) we obtain

$$\begin{aligned} q\big(s_z(x), s_z(y)\big) &= \frac{|s_z(x) - s_z(y)|}{\sqrt{1+|s_z(x)|^2}\,\sqrt{1+|s_z(y)|^2}} \\ &= \frac{(1+|z|^2)|x-y|}{\sqrt{|z|^2|x-\tilde{z}|^2+|x-z|^2}\,\sqrt{|z|^2|y-\tilde{z}|^2+|y-z|^2}}. \end{aligned} \tag{1.49}$$

By the Pythagorean theorem $|\pi(x) - \pi(z)|^2 + |\pi(x) - \pi(\tilde{z})|^2 = 1$ or, equivalently, $q(x,z)^2 + q(x,\tilde{z})^2 = 1$ or

$$\frac{|x-z|^2}{(1+|x|^2)(1+|z|^2)} + \frac{|x-\tilde{z}|^2}{(1+|x|^2)(1+|z|^{-2})} = 1.$$

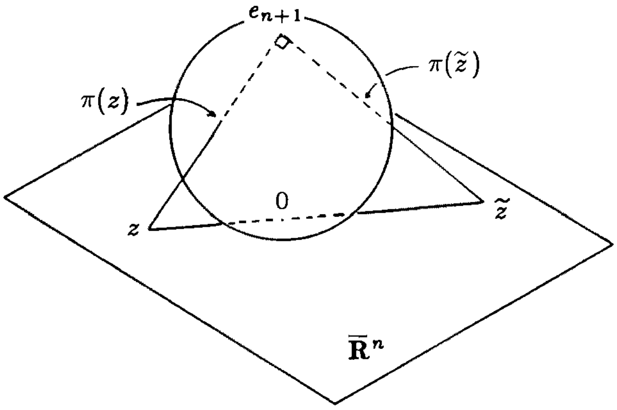


Diagram 1.6.

This yields

$$(1.50)\qquad \begin{aligned} 1+|x|^2 &= \frac{|x-z|^2}{1+|z|^2}+\frac{|x-\tilde{z}|^2}{1+|z|^{-2}} = \frac{|x-z|^2}{1+|z|^2}+\frac{|z|^2|x-\tilde{z}|^2}{1+|z|^2}\ , \\ 1+|y|^2 &= \frac{|y-x|^2}{1+|z|^2}+\frac{|z|^2|y-\tilde{z}|^2}{1+|z|^2}\ . \end{aligned}$$

By substituting (1.50) into (1.49) we obtain

$$q\big(s_z(x), s_z(y)\big) = q(x,y)\ ,$$

showing that s_z preserves the spherical distance between points x, y in $\mathbf{R}^n$. It is left as an exercise for the reader to prove the case when x or y equals ∞. ❐

1.51. Lemma. *A Möbius transformation h is a spherical isometry if and only if $t_{h(0)} \circ h \in \mathcal{O}(n)$.*

Proof. Assume that $h \in \mathcal{GM}$ is a spherical isometry. Then $f = t_{h(0)} \circ h \in \mathcal{GM}(\mathbf{B}^n)$ with $f(0) = 0$, and hence $f \in \mathcal{O}(n)$ by 1.37. The converse implication is trivial. ❐

In some questions it is useful to apply the following *isometric decomposition* of an inversion, which follows from [BE, p. 31, Theorem 3.2.4].

1.52. Lemma. *Let $a \in \mathbf{R}^n$, $r > 0$, and let $b \in \mathbf{R}^n$, $u > 0$, be such that $B^n(a,r) = Q(b,u)$. If f is the inversion in $S^{n-1}(a,r)$, then*

$$f = t_b^{-1} \circ f_1 \circ t_b ,$$

where t_b is the spherical isometry defined in (1.46) and f_1 is the inversion in $S^{n-1}(u/\sqrt{1-u^2}) = \partial Q(0,u)$.

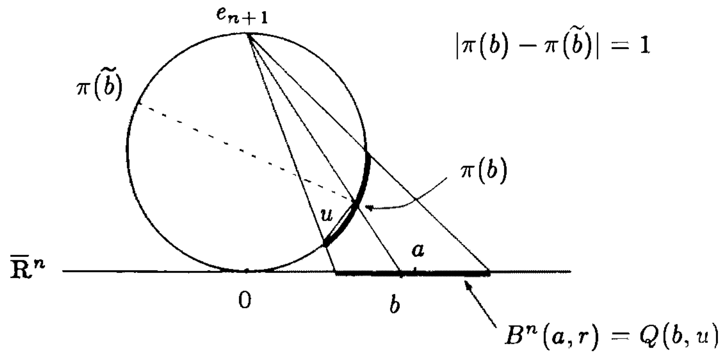


Diagram 1.7.

1.53. Exercise. Show that $B^n(a,r)$ and $B^n(v)$, where $r^2 < 1 + |a|^2$,

$$v = \frac{2r}{\sqrt{(1+(|a|+r)^2)(1+(|a|-r)^2)} + 1 + |a|^2 - r^2}$$

have equal spherical diameters. Note that $v < 1$. Conclusion: The inversion f_1 in 1.52 is in fact the inversion in a euclidean sphere with radius v and center 0.

1.54. Lemma. *Each of the following Möbius transformations is a bilipschitz mapping in the spherical metric with the given constant:*

(1) $f(x) = kx$, $k \geq 1$: $\mathrm{Lip}(f) = k$.

(2) *The inversion in* $S^{n-1}(t)$, $t \in (0,1)$: $\mathrm{Lip}(f) = t^{-2}$.

(3) *The inversion in* $S^{n-1}(a,r)$, $r^2 < 1 + |a|^2$:

$$\mathrm{Lip}(f) = \left(\frac{\sqrt{(1+(|a|+r)^2)(1+(|a|-r)^2)} + 1 + |a|^2 - r^2}{2r} \right)^2 .$$

(4) $f(x) = x + b$: $\mathrm{Lip}(f) = 1 + \frac{1}{2}|b| \left(|b| + \sqrt{4 + |b|^2}\right)$.

Proof. (1) Clearly $fB^n(\frac{1}{k}) = \mathbf{B}^n$. If π_2 is the map in 1.20, then

$$\pi_2^{-1} B^n(\tfrac{1}{k}) = S^n \cap B^n(-e_{n+1}, 2/\sqrt{1+k^2}) = A$$

and $\pi_2 \mathbf{B}^n = S_-^n = \{ x \in S^n : x_{n+1} < 0 \}$.

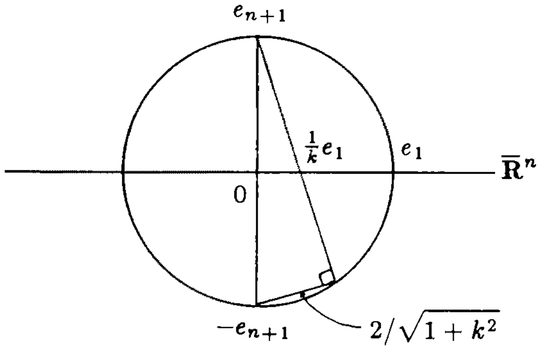


Diagram 1.8.

Hence $\pi_2 \circ f \circ \pi_2^{-1} \colon S^n \to S^n$ maps A onto S^n_-. Let α be the angle between $[0, e_{n+1}]$ and $[e_{n+1}, \frac{1}{k}e_1]$. Obviously $\tan\alpha = \frac{1}{k}$ and the Lipschitz constant of $\pi_2 \circ f \circ \pi_2^{-1}$ in the euclidean metric of $\mathbf{R}^{n+1}$ (restricted to S^n) is the same as the Lipschitz constant of f in the spherical metric, $\mathrm{Lip}(f)$. It follows from 1.42(1) that $\mathrm{Lip}(f) = k$.

(2) Since the proof is similar to the above proof, we indicate only the changes. First f maps $S^{n-1}(t^2)$ onto S^{n-1} (and $B^n(t^2)$ onto $\overline{\mathbf{R}}^n \setminus \mathbf{B}^n$). As above in the proof of part (1) we see that $\mathrm{Lip}(f) = t^{-2}$.

(3) The proof follows from 1.52, 1.53, and part (2).

(4) Again the proof is similar to the one in (1). Observe first that $g = \pi_2 \circ f \circ \pi_2^{-1}$ preserves the 2-dimensional plane containing e_{n+1}, $-e_{n+1}$ and $-b$, and that $g(e_{n+1}) = e_{n+1}$, $g(\pi_2(-b)) = -e_{n+1}$.

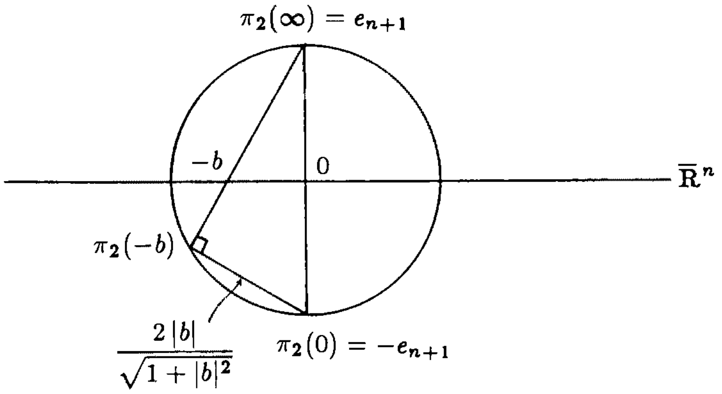


Diagram 1.9.

By 1.37 we see that $g = k \circ T_a$, $k \in \mathcal{O}(n+1)$, $T_a \in \mathcal{GM}(\mathbf{B}^{n+1})$. By elementary geometry $|a| = 1/\sqrt{1+4|b|^{-2}}$, and hence (1.40) yields

$$\mathrm{Lip}(f) = \mathrm{Lip}(g) = \frac{1+|a|}{1-|a|} = \frac{\sqrt{4+|b|^2}+|b|}{\sqrt{4+|b|^2}-|b|} = 1 + \tfrac{1}{2}|b|\left(|b| + \sqrt{4+|b|^2}\right).$$

1.55. Exercise. Let $x, y \in \overline{\mathbf{R}}^n$. Show that $q(x,y) = t$ if and only if there exists a spherical isometry h with $|h(x)| = |h(y)| = 1$ and $|h(x) - h(y)| = 2t$. Prove that the Lipschitz constant of $T_a|\mathbf{B}^n$ in the euclidean metric is equal to the Lipschitz constant of $T_a|\overline{\mathbf{R}}^n$ in the spherical metric.

1.56. Corollary. *Let* $u \in (0, 1/\sqrt{2}\,]$, *let* f *be the inversion in* $S^{n-1}(a,r)$, *and let* $S^{n-1}(a,r) = \partial Q(b,u)$ *for some* $b \in \overline{\mathbf{R}}^n$. *Then* $\mathrm{Lip}(f) = u^{-2} - 1$.

Proof. By 1.52 f and $t_b \circ f \circ t_b^{-1} = g$ have equal Lipschitz constants. By 1.52 g is the inversion in $S^{n-1}(u/\sqrt{1-u^2}) = \partial Q(0,u)$. Hence by 1.54(2) and 1.25(1), $\mathrm{Lip}(f) = u^{-2} - 1$. □

1.57. Exercise. Let x, y, w be three points in $\mathbf{R}^n$. Show that

$$q(x,y)/c \le q(x-w, y-w) \le c\, q(x,y)$$

where $c = \mathrm{Lip}(h)$ and $h(x) = x - w$. [Hint: 1.54(4).]

1.58. Exercise. (Continuation to 1.18(4) and 1.53.) Assume that $x, y, z \in B^n(a,r)$ with $x \neq z$. Show that

$$\frac{1}{c}\frac{|z-y|}{|z-x|} \le \frac{q(z,y)}{q(z,x)} \le c\,\frac{|z-y|}{|z-x|},$$

where c depends only on $q(\overline{B}^n(a,r))$.

1.59. Exercise. Let f be the inversion in $S^{n-1}(r)$, where $r \in (0,1]$. Given an integer $m = 2, 3, \ldots$ show that there are $f_1, \ldots, f_m$ in $\mathcal{GM}$ with the two properties

$$f = f_1 \circ \cdots \circ f_m \quad \text{and} \quad \mathrm{Lip}(f) = \mathrm{Lip}(f_1) \ldots \mathrm{Lip}(f_m).$$

1.60. Remark. As shown e.g. by the Riemann mapping theorem, the class of conformal mappings in the plane is extremely large. According to a deep classical theorem of Liouville the multidimensional case is radically different: If D is a domain in $\mathbf{R}^n$, $n \geq 3$, and $f\colon D \to fD \subset \mathbf{R}^n$ is conformal, then f can be written in the form $f = g|D$ where $g \in \mathcal{GM}(\overline{\mathbf{R}}^n)$. This result was proved by Liouville for C^3-mappings, and a similar result under weaker hypotheses was obtained by F. W. Gehring in 1962 [G2, Theorem 16] and by Yu. G. Reshetnyak in 1967 (see [R13] for more details). A new proof was recently given by B. Bojarski and T. Iwaniec [BI1]. Further generalizations of these results were obtained by Yu. G. Reshetnyak (see [R13] and the references therein). For additional references see [WI, p. 437].

1.61. Notes. A. F. Beardon [BE] has given a thorough account of the theory of Möbius transformations in $\overline{\mathbf{R}}^n$. See also L. V. Ahlfors [A5] and J. B. Wilker [WI]. An illuminating representation of the stereographic projection and the spherical metric is contained in the classical book of D. Hilbert and S. Cohn–Vossen [HCV]. A thorough discussion of two–dimensional Möbius transformations is contained in the books of C. Carathéodory [CA], L. R. Ford [FO], and H. Schwerdtfeger [SC]. The book of M. Berger [BER] contains numerous excellent illustrations related to the topic of this section.

2. Hyperbolic geometry

Hyperbolic geometry can be developed in the context of two spaces or, as they are sometimes called, models. These two models of the hyperbolic space are the unit ball $\mathbf{B}^n$ and the Poincaré half–space

$$\mathbf{H}^n = \mathbf{R}^n_+ = \{ (x_1, \ldots, x_n) \in \mathbf{R}^n : x_n > 0 \} .$$

These two models can be equipped with a hyperbolic metric ρ that is unique up to a multiplicative constant in either model. In either model the metric is normalized (by giving the element of length of the metric) in such a way that for all $x, y \in \mathbf{B}^n$

$$\rho_{\mathbf{H}^n}\big(h(x), h(y)\big) = \rho_{\mathbf{B}^n}(x, y)$$

whenever $h \in \mathcal{GM}$ and $h\mathbf{B}^n = \mathbf{H}^n$. Therefore both models are conformally compatible in the sense that the two metric spaces $(\mathbf{B}^n, \rho)$ and $(\mathbf{H}^n, \rho)$ can be identified. This compatibility is very convenient in computations because we may do a computation in that model in which it is easier, without loss of generality. In what follows we shall use the symbols $\mathbf{R}^n_+$ and $\mathbf{H}^n$ interchangeably.

For $A \subset \mathbf{R}^n$ let $A_+ = \{x \in A : x_n > 0\}$. We define a weight function $w\colon \mathbf{R}^n_+ \to \mathbf{R}_+ = \{x \in \mathbf{R} : x > 0\}$ by

$$(2.1) \qquad w(x) = \frac{1}{x_n}\ , \ x = (x_1, \ldots, x_n) \in \mathbf{R}^n_+ \, .$$

If $\gamma\colon [0,1) \to \mathbf{R}^n_+$ is a continuous mapping such that $\gamma[0,1)$ is a rectifiable curve with length $s = \ell(\gamma)$, then γ has a normal representation $\gamma^0\colon [0,s) \to \mathbf{R}^n_+$ parametrized by arc length (see J. Väisälä [V7, p. 5]). The *hyperbolic length* of $\gamma[0,1)$ is defined by

$$(2.2) \qquad \ell_h(\gamma[0,1)) = \int_0^s \left|(\gamma^0)'(t)\right| w(\gamma^0(t))\, dt = \int_\gamma \frac{|dx|}{x_n}\, .$$

If $A \subset \mathbf{R}^n_+$ is a (Lebesgue) measurable set we define the *hyperbolic volume* of A by

$$(2.3) \qquad m_h(A) = \int_A w(x)^n dm(x)\, ,$$

where m stands for the n-dimensional Lebesgue measure and w is as in (2.1). If $a, b \in \mathbf{R}^n_+$, then the *hyperbolic distance* between a and b is defined by

$$(2.4) \qquad \rho(a,b) = \inf_{\alpha \in \Gamma_{ab}} \ell_h(\alpha) = \inf_{\alpha \in \Gamma_{ab}} \int_\alpha \frac{|dx|}{x_n}\, ,$$

where Γ_{ab} stands for the collection of all rectifiable curves in $\mathbf{R}^n_+$ joining a and b. Sometimes the more complete notation $\rho_{\mathbf{R}^n_+}(a,b)$ or $\rho_{\mathbf{H}^n}(a,b)$ will be employed. The infimum in (2.4) is in fact attained: for given $a, b \in \mathbf{R}^n_+$ there exists a circular arc L perpendicular to $\partial\mathbf{R}^n_+$ such that the closed subarc $J[a,b]$ of L with end points a and b satisfies

$$(2.5) \qquad \rho(a,b) = \ell_h(J[a,b]) = \int\limits_{J[a,b]} \frac{|dx|}{x_n}\, .$$

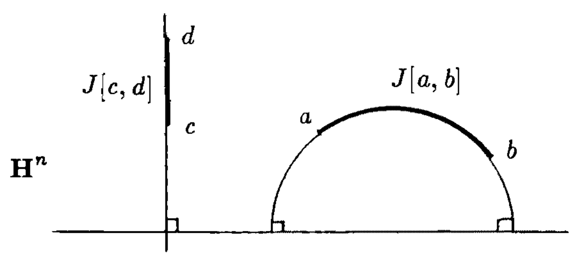


Diagram 2.1.

If a and b are located on a normal of $\partial\mathbf{R}^n_+$, then $J[a,b] = [a,b] = \{(1-t)a + tb : 0 \le t \le 1\}$ (cf. [BE, p. 134]). Because of the (hyperbolic) length–minimizing property (2.5), the arc $J[a,b]$ will be called the *geodesic segment* joining a and b.

Knowing the geodesics, we calculate the hyperbolic distance in two special cases. First, for $r, s > 0$ we obtain

$$\rho(re_n, se_n) = \left| \int_s^r \frac{dt}{t} \right| = \left| \log \frac{r}{s} \right| . \tag{2.6}$$

Second, if $\varphi \in (0, \frac{1}{2}\pi)$ we denote $u_\varphi = (\cos\varphi)e_1 + (\sin\varphi)e_n$ and calculate

$$\rho(e_n, u_\varphi) = \int\limits_{J[u_\varphi, e_n]} \frac{d\alpha}{\sin\alpha} = \int\limits_{\varphi}^{\pi/2} \frac{d\alpha}{\sin\alpha} = \log\cot\tfrac{1}{2}\varphi . \tag{2.7}$$

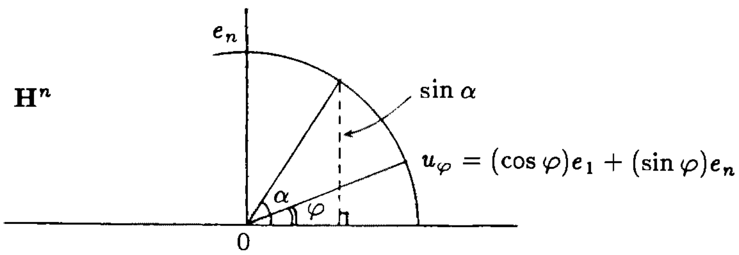


Diagram 2.2.

We shall often make use of the hyperbolic functions $\operatorname{sh} x = \sinh x$, $\operatorname{ch} x = \cosh x$, $\operatorname{th} x = \tanh x$, $\operatorname{cth} x = \coth x$ and their inverse functions which are listed in 2.12. The above formulae (2.6) and (2.7) are special cases of the general formula (see [BE, p. 35])

$$\operatorname{ch}\rho(x,y) = 1 + \frac{|x-y|^2}{2x_n y_n} , \quad x, y \in \mathbf{H}^n = \mathbf{R}^n_+ . \tag{2.8}$$

Note that by this formula the hyperbolic distance $\rho(x,y)$ is completely determined once the euclidean distances $x_n = d(x, \partial\mathbf{H}^n)$, $y_n = d(y, \partial\mathbf{H}^n)$, and $|x-y|$ are known. For another formulation of (2.8) let $z, w \in \mathbf{H}^n$, let L be an arc of a circle perpendicular to $\partial\mathbf{H}^n$ with $z, w \in L$ and let $\{z_*, w_*\} = L \cap \partial\mathbf{H}^n$, the points being labelled so that z_*, z, w, w_* occur in this order on L. Then (cf. [BE, p. 133, (7.26)])

$$\rho(z,w) = \log |\, z_*, z, w, w_* \,| . \tag{2.9}$$

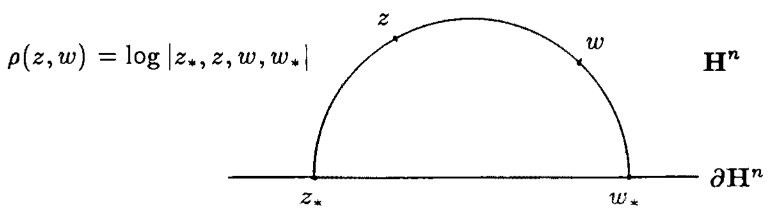


Diagram 2.3.

Note that (2.6) is a special case of (2.9) when $z_* = 0$ and $w_* = \infty$ because $|\,0, z, w, \infty\,| = |w|/|z|$ for $z, w \in \mathbf{H}^n$. The invariance of ρ is apparent by (2.9) and (1.28): Given f in $\mathcal{GM}(\mathbf{H}^n)$ and $x, y \in \mathbf{H}^n$, then

$$\rho(x,y) = \rho(f(x), f(y)) \; . \tag{2.10}$$

For $a \in \mathbf{H}^n$ and $M > 0$ the *hyperbolic ball* $\{\, x \in \mathbf{H}^n : \rho(a,x) < M \,\}$ is denoted by $D(a,M)$. It is well known that $D(a,M) = B^n(z,r)$ for some z and r (this also follows from (2.10)!). This fact together with the observation that $\lambda t e_n,\ (t/\lambda) e_n \in \partial D(te_n, M)$, $\lambda = e^M$ (cf. (2.6)), yields

$$\begin{cases} D(te_n, M) = B^n\big((t \operatorname{ch} M) e_n, t \operatorname{sh} M\big) \; , \\ B^n(te_n, rt) \subset D(te_n, M) \subset B^n(te_n, Rt) \; , \\ r = 1 - e^{-M} \; , \quad R = e^M - 1 \; . \end{cases} \tag{2.11}$$

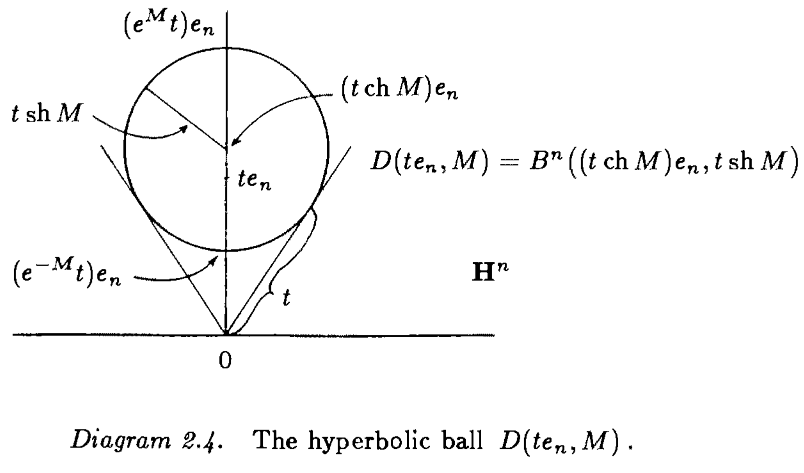


Diagram 2.4. The hyperbolic ball $D(te_n, M)$.

2.12. Remark. The hyperbolic functions $\operatorname{sh} x$, $\operatorname{ch} x$, $\operatorname{th} x$, $\operatorname{cth} x$ and their inverse functions $\operatorname{arsh} x$, $\operatorname{arch} x$, $\operatorname{arth} x$, $\operatorname{arcth} x$ (denoted by some authors as $\sinh^{-1} x$, $\cosh^{-1} x$ etc.) occur often in what follows. Recall that

$$\begin{cases} \operatorname{arsh} x = \log(x + \sqrt{x^2+1}) \ , \ x \geq 0 \ , \\ \operatorname{arch} x = \log(x + \sqrt{x^2-1}) \ , \ x \geq 1 \ , \\ \operatorname{arth} x = \frac{1}{2} \log \dfrac{1+x}{1-x} \ , \ 0 \leq x < 1 \ , \\ \operatorname{arcth} x = \frac{1}{2} \log \dfrac{x+1}{x-1} \ , \ x > 1 \ . \end{cases}$$

For easy reference we record the following inequalities, whose proofs we leave as exercises:

$$\log(1+x) \leq \operatorname{arsh} x \leq 2\log(1+x) \ , \ x \geq 0 \ , \tag{2.13}$$

$$2\log\big(1 + \sqrt{\tfrac{1}{2}(x-1)}\,\big) \leq \operatorname{arch} x \leq 2\log\big(1 + \sqrt{2(x-1)}\,\big) \ , \ x \geq 1 \ . \tag{2.14}$$

So far we have discussed only the hyperbolic geometry of $\mathbf{H}^n = \mathbf{R}^n_+$. Now we are going to give the corresponding formulae for $\mathbf{B}^n$. The weight function $w\colon \mathbf{B}^n \to \mathbf{R}_+$ is now defined by

$$w(x) = \frac{2}{1-|x|^2} \ , \ x \in \mathbf{B}^n \ , \tag{2.15}$$

(cf. (2.1)). The *hyperbolic distance* between a and b in $\mathbf{B}^n$, denoted by $\rho_{\mathbf{B}^n}(a,b) = \rho(a,b)$, is defined by a formula analogous to (2.5) ; the same is true about the *hyperbolic volume* of a measurable set $A \subset \mathbf{B}^n$. For $a, b \in \mathbf{B}^n$ the *geodesic segment* $J[a,b]$ joining a to b is an arc of a circle orthogonal to S^{n-1}. In a limiting case the points a and b are located on a euclidean line through 0.

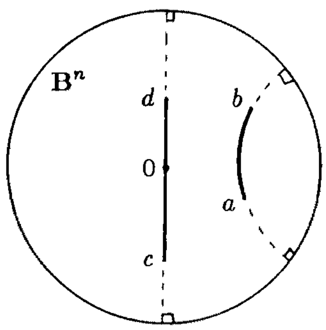


Diagram 2.5.

In particular, $J[0,te_1] = [0,te_1]$ for $0 < t < 1$ and we have

$$\rho(0,te_1) = \int\limits_{[0,te_1]} \frac{2|dx|}{1-|x|^2} = \int\limits_0^t \frac{2ds}{1-s^2} = \log\frac{1+t}{1-t} = 2\operatorname{arth} t\,. \tag{2.16}$$

It follows from (2.16) that for $s \in (-t,t)$

$$\rho(se_1,te_1) = \log\Big(\frac{1+t}{1-t}\cdot\frac{1-s}{1+s}\Big)\,. \tag{2.17}$$

A counterpart of (2.8) for $\mathbf{B}^n$ is

$$\operatorname{sh}^2\big(\tfrac{1}{2}\rho(x,y)\big) = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}\,,\quad x,y\in\mathbf{B}^n\,, \tag{2.18}$$

(cf. [BE, p. 40]). As in the case of $\mathbf{H}^n$, we see by (2.18) that the hyperbolic distance $\rho(x,y)$ between x and y is completely determined by the euclidean quantities $|x-y|$, $d(x,\partial\mathbf{B}^n)$, $d(y,\partial\mathbf{B}^n)$. Finally, we have also

$$\rho(x,y) = \log|\,x_*,x,y,y_*|\,, \tag{2.19}$$

where x_*, y_* are defined as in (2.9): If L is the circle orthogonal to S^{n-1} with $x,y\in L$, then $\{x_*,\,y_*\} = L\cap S^{n-1}$, the points being labelled so that x_*, x, y, y_* occur in this order on L. It follows from (2.19) and (1.28) that

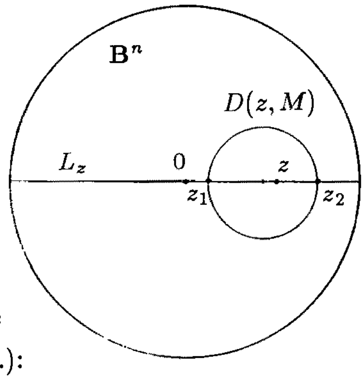
$$\rho(x,y) = \rho(h(x),h(y)) \tag{2.20}$$

for all $x,y\in\mathbf{B}^n$ whenever h is in $\mathcal{GM}(\mathbf{B}^n)$. Finally, in view of (1.28), (2.9), and (2.19) we have

$$\rho_{\mathbf{B}^n}(x,y) = \rho_{\mathbf{H}^n}(g(x),g(y))\,,\quad x,y\in\mathbf{B}^n\,, \tag{2.21}$$

whenever g is a Möbius transformation with $g\mathbf{B}^n = \mathbf{H}^n$.

It is well known that the balls $D(z,M)$ of $(\mathbf{B}^n,\rho)$ are balls in the euclidean geometry as well, i.e. $D(z,M) = B^n(y,r)$ for some $y\in\mathbf{B}^n$ and $r>0$. Making use of this fact, we shall find y and r. Let L_z be a euclidean line through 0 and z and $\{z_1,z_2\} = L_z\cap\partial D(z,M)$, $|z_1|\le|z_2|$. We may assume that $z\neq 0$ since with obvious changes the following argument works for $z=0$ as well. Let $e = z/|z|$ and $z_1 = se$, $z_2 = ue$, $u\in(0,1)$, $s\in(-u,u)$. It follows from (2.17) that



$$\rho(z_1,z) = \log\Big(\frac{1+|z|}{1-|z|}\cdot\frac{1-s}{1+s}\Big) = M\,,$$

$$\rho(z_2,z) = \log\Big(\frac{1+u}{1-u}\cdot\frac{1-|z|}{1+|z|}\Big) = M\,.$$

Solving these for s and u and using the fact that $D(z,M) = B^n\big(\frac{1}{2}(z_1+z_2),\ \frac{1}{2}|u-s|\big)$ one obtains the following formulae (Exercise: Verify the computation.):

Diagram 2.6.

$$(2.22)\qquad \begin{cases} D(x,M) = B^n(y,r)\,, \\ y = \dfrac{x(1-t^2)}{1-|x|^2t^2}\,,\quad r = \dfrac{(1-|x|^2)t}{1-|x|^2t^2}\,,\quad t = \operatorname{th}\frac{1}{2}M\,, \end{cases}$$

and

$$(2.23)\qquad \begin{cases} B^n\big(x,\ a(1-|x|)\big) \subset D(x,M) \subset B^n\big(x,\ A(1-|x|)\big)\,, \\ a = \dfrac{t(1+|x|)}{1+|x|t}\,,\quad A = \dfrac{t(1+|x|)}{1-|x|t}\,,\quad t = \operatorname{th}\frac{1}{2}M\,. \end{cases}$$

We shall often need a special case of (2.22):

$$(2.24)\qquad D(0,M) = B^n(\operatorname{th}\tfrac{1}{2}M)\,.$$

A standard application of formula (2.24) is the following observation. Let T_x be in $\mathcal{M}(\mathbf{B}^n)$ as defined in 1.34 with $T_x(x) = 0$. Fix $x,y\in\mathbf{B}^n$ and $z\in J[x,y]$ with $\rho(z,x) = \rho(z,y) = \frac{1}{2}\rho(x,y)$. Then $T_z(x) = -T_z(y)$ and (2.24) yields

$$(2.25)\qquad \begin{cases} |T_x(y)| = \operatorname{th}\frac{1}{2}\rho(x,y)\,, \\ |T_z(x)| = \operatorname{th}\frac{1}{4}\rho(x,y)\,. \end{cases}$$

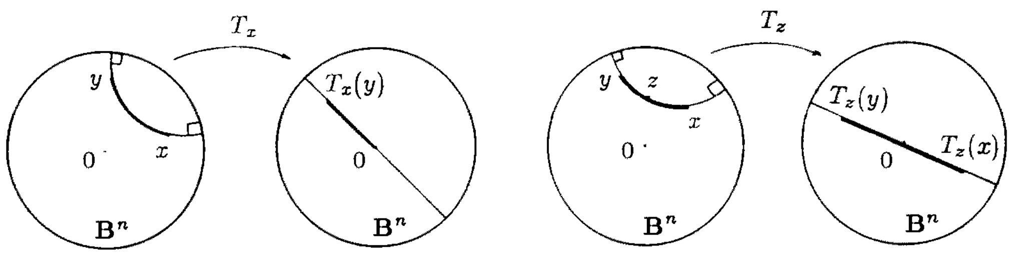


Diagram 2.7.

We next derive a few inequalities from (2.23). By studying the expression for the radius r in (2.23) we see that

$$d(\overline{D}(z,M)) \le d(\overline{D}(0,M)) = 2\,\mathrm{th}\,\tfrac{1}{2}M \tag{2.26}$$

for all $z \in \mathbf{B}^n$ and all $M > 0$. This yields a sharp inequality between the euclidean and hyperbolic distances as follows. For given $x, y \in \mathbf{B}^n$ choose $z \in J[x,y]$ with $\rho(x,z) = \rho(z,y) = \frac{1}{2}\rho(x,y)$. Then with $M = \frac{1}{2}\rho(x,y)$ (2.26) yields the useful inequality

$$|x-y| \le 2\,\mathrm{th}\,\tfrac{1}{4}\rho(x,y)\ . \tag{2.27}$$

Equality holds here if $x = -y$. Because $\mathrm{th}\, A \le A$, (2.27) yields also the crude estimate

$$|x-y| \le \tfrac{1}{2}\rho(x,y) \tag{2.28}$$

for $x, y \in \mathbf{B}^n$.

2.29. Exercise. Verify the following elementary relations.

(1) $1 - e^{-s} \le \mathrm{th}\, s \le 1 - e^{-2s}$ for $s \ge 0$.

(2) If $s \ge 0$, then

$$\mathrm{th}\, s = \frac{\mathrm{th}\, 2s}{1 + \sqrt{1 - \mathrm{th}^2\, 2s}}\ .$$

Further, if $u \in [0,1]$ and $2s = \mathrm{arth}\, u$, then

$$\mathrm{th}\, s = \frac{u}{1+\sqrt{1-u^2}} \le \tfrac{1}{2}(u+u^2)\ .$$

(3) $\log \mathrm{th}\, s = -2\,\mathrm{arth}\, e^{-2s}$, $s > 0$.

(4) $\dfrac{x}{1+x} \le 1 - e^{-x} \le x$, for $x > -1$ [AS, 4.2.32].

(5) Show that

$$\frac{1+\mathrm{th}\, px}{1-\mathrm{th}\, px} = \left(\frac{1+\mathrm{th}\, x}{1-\mathrm{th}\, x}\right)^p$$

for $p = 1, 2, \ldots$ and $x > 0$.

2.30. Exercise. Show that for $x,\ y \in \mathbf{B}^n$

$$\rho(x,y) \le \frac{2|x-y|}{\min\{\,1-|x|,\ 1-|y|\,\}}\ .$$

2.31. Exercise. Applying (2.23) show that if $D(x,M) = B^n(y,r)$, then r admits an estimate

$$(1-|y|)\,b \le r \le (1-|y|)\,B$$

where b and B depend only on M. Show also that the numbers a and A in (2.23) have lower and upper bounds depending only on M. In particular, A/a has an upper bound depending only on M.

2.32. Exercise. Let $x_0 \in \mathbf{B}^n$, $M > 0$ and

$$v = \min\{\,|z-x_0| : \rho(x_0,z) = M\,\}\,,$$
$$V = \max\{\,|z-x_0| : \rho(x_0,z) = M\,\}\,.$$

Find an upper bound for V/v by applying 1.43(1).

2.33. Exercise. Rewrite (2.8) and (2.19) using the identity $2\,\mathrm{sh}^2 A = \mathrm{ch}\,2A - 1$.

Given distinct points x and y in $\mathbf{B}^n$ or $\mathbf{H}^n$ one can express the Poincaré distance $\rho(x,y)$ in terms of the absolute ratio $|x_*,x,y,y_*|$ by virtue of the formulae (2.9) and (2.19) where x_* and y_* are the "end–points" of a geodesic segment containing x and y. Sometimes it will be convenient to express $\rho(x,y)$ in a different way without refering to the points x_* and y_* at all. Such an expression can be achieved by exploiting an extremal property of $\rho(x,y)$ as we shall show in the next section (see also Section 8).

The formulae (2.8) and (2.18), which give explicit expressions for $\rho_{\mathbf{H}^n}(x,y)$ and $\rho_{\mathbf{B}^n}(x,y)$, respectively, are of fundamental importance for hyperbolic geometry. As a matter of fact, many formulae of this section can be derived directly from these formulae. For many applications it would be formally adequate to define the hyperbolic distance in terms of (2.8) and (2.18) without any reference to the geometric interpretation involving elements of lengths or the length–minimizing property of geodesics. These geometric notions and their invariance properties are, however, the reason why the hyperbolic metric is so useful and natural in many applications.

The reader may show as an exercise that (2.23) follows from (2.18). The explicit expressions (2.8) and (2.18) for $\rho(x,y)$ are somewhat complicated. Often it will be sufficient to give bounds for $\rho(x,y)$ in terms of simple comparison functions. We now introduce such a function.

For an open set D in $\mathbf{R}^n$, $D \neq \mathbf{R}^n$, define $d(z) = d(z, \partial D)$ for $z \in D$ and

$$j_D(x,y) = \log\Big(1 + \frac{|x-y|}{\min\{d(x),\, d(y)\}}\Big) \tag{2.34}$$

for $x, y \in D$. If $A \subset D$ is non-empty define

$$j_D(A) = \sup\{\, j_D(x,y) : x, y \in A \,\} . \tag{2.35}$$

An elementary (but lengthy) argument shows that $j_D(x,y)$ is a metric on D.

2.36. Lemma. *The following inequalities*

(1) $j_D(x,y) \ge \Big|\log \dfrac{d(x)}{d(y)}\Big|$,

(2) $j_D(x,y) \le \Big|\log \dfrac{d(x)}{d(y)}\Big| + \log\Big(1 + \dfrac{|x-y|}{d(x)}\Big) \le 2\, j_D(x,y)$

hold for all $x, y \in D$.

Proof. (1) The proof follows because $d(y) \le d(x) + |x-y|$.

(2) If $d(x) \le d(y)$, the proof is obvious in view of (2.34). If $d(x) > d(y)$,

$$\begin{aligned} j_D(x,y) &= \log\Big(1 + \frac{|x-y|}{d(y)}\Big) \le \log\Big(\frac{d(x)}{d(y)} + \frac{d(x)}{d(y)}\frac{|x-y|}{d(y)}\Big) \\ &= \log\frac{d(x)}{d(y)} + \log\Big(1 + \frac{|x-y|}{d(y)}\Big) \le 2\, j_D(x,y) \ , \end{aligned}$$

where in the last step the inequality in part (1) was applied. □

2.37. Exercise. For an open set $D \subset \mathbf{R}^n$ with $D \neq \mathbf{R}^n$ and for a non-empty set A in D with $d(A, \partial D) > 0$ put

$$r_D(A) = \frac{d(A)}{d(A, \partial D)} .$$

Show that

$$\tfrac{1}{2}\log\big(1 + r_D(A)\big) \le \log\big(1 + \tfrac{1}{2} r_D(A)\big) \le j_D(A) \le \log\big(1 + r_D(A)\big) .$$

2.38. Exercise. Let G and G' be proper subdomains of $\mathbf{R}^n$ with $G' \subset G$. Show that $j_G(x,y) \le j_{G'}(x,y)$ for all $x, y \in G'$. For $w \in \mathbf{R}^n$ set $R_w = \mathbf{R}^n \setminus \{w\}$ and define

$$h_G(x,y) = \sup\{\, j_{R_w}(x,y) : w \in \partial G \,\}$$

for all $x, y \in G$. Show that if $w \in \partial G$

$$j_G(x,y) = h_G(x,y) \ge \left| \log \frac{|y-w|}{|x-w|} \right| ; \quad x,\, y \in G . \tag{2.39}$$

Moreover, if $d(x) \le d(y)$ and $z \in \partial G$ with $|x-z| = d(x)$ prove that

$$\frac{|y-z|}{|x-z|} \ge \tfrac{1}{2}\big(\exp j_G(x,y) - 1\big) .$$

2.40. Exercise. (1) Let $B = S^{n-1}(e_n, 1) \cap \{ x \in \mathbf{H}^n \, : \, x_n \ge 1 \}$. Find $\max\{ \rho_{\mathbf{H}^n}(e_n, x) \, : \, x \in B\}$ and $\min\{ \rho_{\mathbf{H}^n}(e_n, x) \, : \, x \in B\}$.

(2) For an open set $D \subset \mathbf{R}^n$, $D \ne \mathbf{R}^n$, define

$$\tilde{j}_D(x,y) = \log\Big[\Big(1 + \frac{|x-y|}{d(x)}\Big)\Big(1 + \frac{|x-y|}{d(y)}\Big)\Big] .$$

Show that for all $x, y \in D$

$$j_D(x,y) \le \tilde{j}_D(x,y) \le 2\, j_D(x,y) .$$

In the next lemma we show that j_D yields simple two-sided estimates for ρ_D both when $D = \mathbf{B}^n$ and when $D = \mathbf{H}^n$.

2.41. Lemma. (1) $j_{\mathbf{B}^n}(x,y) \le \rho_{\mathbf{B}^n}(x,y) \le 4\, j_{\mathbf{B}^n}(x,y)$ *for* $x, y \in \mathbf{B}^n$.
(2) $j_{\mathbf{H}^n}(x,y) \le \rho_{\mathbf{H}^n}(x,y) \le 2\, j_{\mathbf{H}^n}(x,y)$ *for* $x, y \in \mathbf{H}^n$.

Proof. (1) By (2.19)

$$\mathrm{sh}^2\big(\tfrac{1}{2}\rho_{\mathbf{B}^n}(x,y)\big) = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)} = t^2 \le \Big(\frac{|x-y|}{\min\{1-|x|, 1-|y|\}}\Big)^2$$

and hence by (2.14)

$$\rho_{\mathbf{B}^n}(x,y) \le 4 \log(1+t) \le 4\, j_{\mathbf{B}^n}(x,y) .$$

For the proof of the lower bound we may assume that $|x| \ge |y|$ and $|x| > 0$. Let L be a euclidean line through 0 and x and fix $y' \in \overline{B}^n(|x|) \cap L$ such that $|x-y'| = |x-y|$. Because $|y'| \le |y|$ it follows from (2.19) and (2.18) that

$$\rho_{\mathbf{B}^n}(x,y) \ge \rho_{\mathbf{B}^n}(x,y') \ge \log\Big(\frac{1+|x|}{1-|x|} \cdot \frac{1-|x|+|x-y|}{1+|x|-|x-y|}\Big) \ge j_{\mathbf{B}^n}(x,y) .$$

(2) Denote $u = 1 + |x-y|^2/(2x_n y_n)$. By (2.8) and (2.14) we get

$$\rho_{\mathbf{H}^n}(x,y) \le 2\log(1+\sqrt{2(u-1)}\,) = 2\log\Big(1+\frac{|x-y|}{\sqrt{x_n y_n}}\Big) \le 2j_{\mathbf{H}^n}(x,y)\ .$$

For the proof of the lower bound we may assume that $x_n \le y_n$ and $x = x_n e_n$. Let $y' = (x_n + |x-y|)e_n$. Because $(y')_n \ge y_n$ it follows from (2.8) and (2.6) that

$$\rho_{\mathbf{H}^n}(x,y) \ge \rho_{\mathbf{H}^n}(x,y') \ge \log\Big(1+\frac{|x-y|}{x_n}\Big) = j_{\mathbf{H}^n}(x,y)\ . \quad \Box$$

2.42. Exercise. Solve 1.41(2) with the help of the hyperbolic metric. [Hint: Because of (2.17) the requirement that $\rho(0,a) = \frac{1}{2}\rho(0,re_1)$ leads to $\big(\frac{1+|a|}{1-|a|}\big)^2 = \frac{1+r}{1-r}$, i.e. $|a| = r/(1+\sqrt{1-r^2}\,)$.]

2.43. Exercise. For an open set D in $\mathbf{R}^n$, $D \ne \mathbf{R}^n$, let

$$\varphi_D(x,y) = \log\Big(1+\max\Big\{\frac{|x-y|}{\sqrt{d(x)d(y)}}\,,\ \frac{|x-y|^2}{d(x)d(y)}\Big\}\Big)\ ;\ x,y\in D\ .$$

Show that $j_D(x,y) \le \varphi_D(x,y) \le 2\,j_D(x,y)$. (See also 3.30.)

2.44. Exercise. (1) Observe first that, for $t\in(0,1)$,

$$\rho_{\mathbf{H}^n}(te_n,e_n) = \rho_{\mathbf{H}^n}\big(te_n,\,S^{n-1}(\tfrac{1}{2}e_n,\,\tfrac{1}{2})\big)$$

(cf. (2.8)). Making use of this observation and (2.11) show that

$$B^n(\tfrac{1}{2}e_n,\tfrac{1}{2}) = \bigcup_{t\in(0,1)} D(te_n,\,\log\frac{1}{t})\ .$$

(2) For $p>0$ and $t>0$ let $A(t) = \rho_{\mathbf{H}^n}\big((0,t^p),(t,t^p)\big)$. Find the limits $\lim_{t\to 0}A(t)$ and $\lim_{t\to\infty}A(t)$ in the three cases $p<1$, $p=1$, and $p>1$.

2.45. Exercise. The stereographic projection π_2 (see 1.20) provides a connection between the hyperbolic geometries $(\mathbf{B}^n,\rho)$ and $(\mathbf{R}^{n+1}_-,\rho_-)$ and the spherical geometry of $(\overline{\mathbf{R}}^n,q)$. Verify that $\rho(0,ae_1) = \rho_-\big(\pi_2(0),\pi_2(ae_1)\big)$, $a\in(0,1)$, by computing the absolute ratios $|-e_1,0,ae_1,e_1|$ and $|\pi_2(-e_1),\pi_2(0),\pi_2(ae_1),\pi_2(e_1)|$ (see (2.9) and (2.20)). Note that $2q(0,ae_1) = |\pi_2(0)-\pi_2(ae_1)|$. Let be_1 be the orthogonal projection of $\pi_2(ae_1)$ onto the x_1-axis. Show that $\rho(0,be_1) = 2\rho(0,ae_1)$. [Hint: See the diagram 1.5 in 1.41(2).]

2.46. Exercise. (Continuation of 2.45.) Show that $\pi_2(ae_1) \in S^n \cap S^n(x,r)$ where $S^n(x,r)$ is a sphere orthogonal to S^n with $ae_1 \in S^n(x,r)$. Find x and r.

2.47. Exercise. Let $x,y \in \mathbf{B}^n$ and let $T_x \in \mathcal{M}(\mathbf{B}^n)$ be as defined in 1.34. Show that

$$|T_x y| = \frac{|x-y|}{\sqrt{|x-y|^2 + (1-|x|^2)(1-|y|^2)}} = \frac{s}{\sqrt{1+s^2}} ,$$

where $s^2 = |x-y|^2/((1-|x|^2)(1-|y|^2))$. [Hint: By (2.25) and (2.19)

$$|T_x y|^2 = \operatorname{th}^2\left(\tfrac{1}{2}\rho(x,y)\right) = 1 - \frac{1}{\operatorname{ch}^2(\frac{1}{2}\rho(x,y))} = \frac{s^2}{1+s^2} .]$$

Next let $z \in J[x,y]$ be the hyperbolic midpoint of $J[x,y]$ as in (2.25). Show that

$$\begin{aligned} |T_z x| = |T_z y| &= \frac{s}{1+\sqrt{1-s^2}} \\ &= \frac{|x-y|}{\sqrt{|x-y|^2 + (1-|x|^2)(1-|y|^2)} + \sqrt{(1-|x|^2)(1-|y|^2)}} , \end{aligned}$$

where s is as above. [Hint: Because $\operatorname{th} A = t/(1+\sqrt{1-t^2})$, $t = \operatorname{th} 2A$, one can apply (2.25) and the above computation.] *Moral:* Instead of using these lengthy expressions for $|T_x y|$ and $|T_z y|$ involving euclidean distances it will often be more convenient to use the equivalent formula (2.25) involving the hyperbolic distance $\rho(x,y)$.

2.48. Exercise. Let $x,y \in \mathbf{R}^n$ and let t_x be a spherical isometry as defined in (1.46). Show that

$$|t_x y| = \frac{|x-y|}{\sqrt{(1+|x|^2)(1+|y|^2) - |x-y|^2}} .$$

[Hint: This follows immediately from (1.47) and (1.15).] Let $\alpha \in [0, \frac{1}{2}\pi]$ be such that $\sin\alpha = q(x,y)$. Then α is the angle between the segments $[e_{n+1}, t_x x] = [e_{n+1}, 0]$ and $[e_{n+1}, t_x y]$ at e_{n+1} (see (1.13).) Show that the above formula can be rewritten as

$$|t_x y| = \tan\alpha .$$

Note the analogy with (2.25).

2.49. Exercise. Show that

$$\frac{|f(x)-f(y)|^2}{(1-|f(x)|^2)\,(1-|f(y)|^2)} = \frac{|x-y|^2}{(1-|x|^2)\,(1-|y|^2)}$$

for all f in $\mathcal{GM}(\mathbf{B}^n)$ and all $x,y \in \mathbf{B}^n$. [Hint: Apply (2.19) and (2.21).]

2.50. Exercise. Let $0 < t < 1$ and $f \in \mathcal{GM}(\mathbf{B}^n)$. Show that

$$|f(x) - f(y)| \le \frac{|x-y|}{1-t^2}$$

for $|x|, |y| \le t$. [Hint: Apply 2.49.]

2.51. Remark. The inequality (2.27) together with the formulae (2.25) and 2.47 yield for $x, y \in \mathbf{B}^n$

$$|x-y| \le 2\,\mathrm{th}\,\tfrac{1}{4}\rho(x,y) = \frac{2|x-y|}{\sqrt{|x-y|^2+b^2}\;+b}$$

where $b = \sqrt{(1-|x|^2)(1-|y|^2)}$.

2.52. Exercise (Contributed by M. K. Vamanamurthy). Starting with the identity (cf. 2.47)

$$\mathrm{th}\,\tfrac{1}{2}\rho(x,y) = \frac{|x-y|}{\sqrt{|x-y|^2+(1-|x|^2)(1-|y|^2)}}$$

for $x, y \in \mathbf{B}^n$ verify the following inequalities

$$\text{(1)} \qquad \frac{|x-y|}{1+|x||y|} \le \mathrm{th}\,\tfrac{1}{2}\rho(x,y) \le \frac{|x-y|}{1-|x||y|}\,,$$

$$\text{(2)} \qquad \frac{|x|-|y|}{1-|x||y|} \le \mathrm{th}\,\tfrac{1}{2}\rho(x,y) \le \frac{|x+y|}{1+|x||y|}\,,$$

$$\text{(3)} \qquad \tfrac{1}{2}|x-y| \le \frac{|x-y|}{1+|x||y|+|x|'|y|'} \le \mathrm{th}\,\tfrac{1}{4}\rho(x,y)$$

$$\le \frac{|x-y|}{1-|x||y|+|x|'|y|'} \le \frac{|x-y|}{2(1-\max\{|x|^2,|y|^2\})}\,,$$

where $|x|' = \sqrt{1-|x|^2}$. Can you find similar inequalities for the spherical chordal metric? [Hint: 2.48.]

2.53. Notes. The main source for this section is [BE] and the other references given at the end of Section 1. See also [T, pp. 508-514] and [RE].

3. Quasihyperbolic geometry

In an arbitrary proper subdomain D of $\mathbf{R}^n$ one can define a metric, the quasihyperbolic metric of D, which shares some properties of the hyperbolic metric of $\mathbf{B}^n$ or $\mathbf{H}^n$. We shall now give the definition of the quasihyperbolic metric and state without proof some of its basic properties which we require later on. The quasihyperbolic metric has been systematically developed and applied by F. W. Gehring and his collaborators.

Throughout this section D will denote a proper subdomain of $\mathbf{R}^n$. In D we define a weight function $w\colon D \to \mathbf{R}_+$ by

$$w(x) = \frac{1}{d(x, \partial D)}\ ;\ x \in D\ . \tag{3.1}$$

Using this weight function one defines the *quasihyperbolic length* $\ell_q(\gamma) = \ell_q^D(\gamma)$ of a rectifiable curve γ by a formula similar to (2.2). The *quasihyperbolic distance* between x and y in D is defined by

$$k_D(x,y) = \inf_{\alpha \in \Gamma_{xy}} \ell_q^D(\alpha) = \inf_{\alpha \in \Gamma_{xy}} \int_\alpha w(x)|dx|\ , \tag{3.2}$$

where Γ_{xy} is as in (2.4). It is clear that k_D is a metric on D. It follows from (3.2) that k_D is invariant under translations, stretchings, and orthogonal mappings. (As in (2.3) one can define the *quasihyperbolic volume* of a (Lebesgue) measurable set $A \subset D$, but we shall not make use of this notion.) Given $x, y \in D$ there exists a *geodesic segment* $J_D[x,y]$ of the metric k_D joining x and y (cf. [GO]). However, very little is known about the structure of such geodesic segments $J_D[x,y]$ when D is given. Some regularity properties of geodesic segments have been obtained by G. Martin [MA].

3.3. Remarks. Clearly, $k_{\mathbf{H}^n} = \rho_{\mathbf{H}^n}$, and we see easily that $\rho_{\mathbf{B}^n} \le 2\,k_{\mathbf{B}^n} \le 2\,\rho_{\mathbf{B}^n}$ (cf. (3.1),(2.15)). Hence, the geodesics of $(\mathbf{H}^n, k_{\mathbf{H}^n})$ are those of $(\mathbf{H}^n, \rho_{\mathbf{H}^n})$, but it is a difficult task to find the geodesics of k_D when D is given. The following monotone property of k_D is clear: if D and D' are domains with $D' \subset D$ and $x, y \in D'$, then $k_{D'}(x,y) \ge k_D(x,y)$.

In order to find some estimates for $k_D(x,y)$ we shall employ, as in the case of $\mathbf{H}^n$ and $\mathbf{B}^n$, the metric j_D defined in (2.34). The metric j_D is indeed a natural choice for such a comparison function since both k_D and j_D are invariant under translations, stretchings and orthogonal mappings. A useful inequality is ([GP, Lemma 2.1])

$$k_D(x,y) \ge j_D(x,y) \; ; \; x,\, y \in D \, . \tag{3.4}$$

In combination with 2.36 , (3.4) yields

$$k_D(x,y) \ge \left| \log \frac{d(x)}{d(y)} \right| , \quad d(z) = d(z, \partial D) \, . \tag{3.5}$$

For easy reference we record Bernoulli's inequality

$$\log(1+as) \le a \log(1+s) \; ; \; a \ge 1 \, , \; s > 0 \, . \tag{3.6}$$

3.7. Lemma. (1) *If* $x \in D$, $y \in B_x = B^n(x, d(x))$, *then*

$$k_D(x,y) \le \log\Big(1 + \frac{|x-y|}{d(x) - |x-y|}\Big) \, .$$

(2) *If* $s \in (0,1)$ *and* $|x-y| \le s\, d(x)$, *then*

$$k_D(x,y) \le \frac{1}{1-s}\, j_D(x,y) \, .$$

Proof. (1) Select $z \in \partial B_x$ such that $y \in [x,z]$.

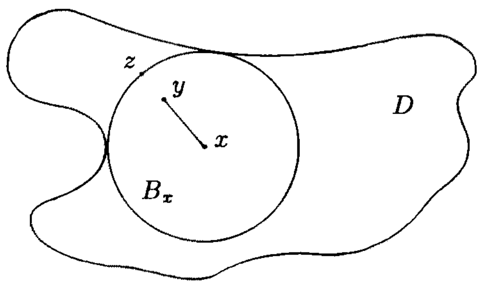


Diagram 3.1.

Because $[x,y] \in \Gamma_{xy}$, from 3.3 we obtain

$$k_D(x,y) \le k_{B_x}(x,y) \le \int\limits_{[x,y]} \frac{|dw|}{d(w)} \le \int\limits_{[x,y]} \frac{|dw|}{|w-z|} = \int\limits_{d(x)-|x-y|}^{d(x)} \frac{dt}{t}$$

$$= \log \frac{d(x)}{d(x) - |x-y|} = \log\Big(1 + \frac{|x-y|}{d(x)-|x-y|}\Big)$$

$$\Big(= j_{\mathbf{R}^n \setminus \{z\}}(x,y) \Big) \, .$$

(2) For the proof of (2) we apply part (1), Bernoulli's inequality (3.6), and the definition of j_D to obtain

$$\begin{aligned} k_D(x,y) &\le \log\Big(1+\frac{|x-y|}{(1-s)d(x)}\Big) \\ &\le \frac{1}{1-s}\log\Big(1+\frac{|x-y|}{d(x)}\Big) \le \frac{1}{1-s}\, j_D(x,y) \end{aligned}$$

as desired. $\square$

We know by 2.41 and 3.3 that if $D = \mathbf{B}^n$, then an inequality similar to 3.7(2) holds for all $x, y \in D$. For a general domain D this is not true, i.e. the ratio

$$A(D) = \sup\Big\{ \frac{k_D(x,y)}{j_D(x,y)} \; : \; x,y \in D\,,\; x \ne y \Big\}$$

may be infinite. For instance, $A(\mathbf{B}^2 \setminus [0,e_1)) = \infty$. (For details, see 3.14.)

3.8. Definition. A domain G in $\mathbf{R}^n$, $G \ne \mathbf{R}^n$, is called *uniform*, if there exists a number $A = A(G) \ge 1$ such that $k_G(x,y) \le A\, j_G(x,y)$ for all $x, y \in G$.

By 2.41 the unit ball $\mathbf{B}^n$ and the half-space $\mathbf{H}^n = \mathbf{R}^n_+$ are uniform domains with the constants 4 and 2, respectively. It follows from the definition that the class of uniform domains is invariant under translations, stretchings and orthogonal maps. It is not difficult to show that the image of a uniform domain under a bilipschitz mapping is again uniform.

Next we shall study the *quasihyperbolic balls* $D_G(x, M) = \{ z \in G : k_G(x,z) < M \}$ when $x \in G$ and $M > 0$. It follows from (3.5) that

$$e^{-M} d(x) \le d(z) \le e^{M} d(x)$$

holds for $z \in \overline{D}_G(x,M)$. Next, for $z \in B^n(x,\,(1-e^{-M})d(x))$ we deduce by 3.7(1) that $k_G(x,z) < M$ and for $z \in \mathbf{R}^n \setminus B^n(x,\,(e^M - 1)d(x))$ we find by (3.4) and (3.5) that $k_G(x,z) > M$. In conclusion, we have proved that

$$\text{(3.9)} \qquad \begin{cases} B^n(x,\, rd(x)) \subset D_G(x,M) \subset B^n(x,\, Rd(x))\,, \\ \overline{D}_G(x,M) \subset \{ z \in G \,:\, e^{-M}d(x) \le d(z) \le e^{M} d(x) \}\,, \\ r = 1 - e^{-M}\,,\;\; R = e^{M} - 1\,. \end{cases}$$

For $G = \mathbf{H}^n$ one can show that the numbers r and R are the best possible (see (2.11)). We shall write $D(x,M)$ for $D_G(x,M)$ if there is no danger of confusion.

Möbius transformations are hyperbolic isometries. That is, if each one of the domains G, G' is a ball or a half-space in $\mathbf{R}^n$ and if f is a Möbius transformation mapping G onto G', then

$$\rho_G(x,y) = \rho_{G'}\big(f(x), f(y)\big)$$

for $x, y \in G$ (cf. (2.9), (2.19), (2.21)). Although the metric k_G does not have this invariance property it is not changed by more than the factor 2 under Möbius transformations. (For a related Möbius invariant metric see J. Ferrand [FE].)

3.10. Lemma. *If G and G' are proper subdomains of $\mathbf{R}^n$ and if f is a Möbius transformation of G onto G', then*

$$\tfrac{1}{2}\, k_G(x,y) \le k_{G'}\big(f(x), f(y)\big) \le 2\, k_G(x,y)$$

for all $x, y \in G$.

A proof of Lemma 3.10 was given by Gehring and Palka in [GP] where also a generalization to the case of quasiconformal mappings was obtained. This generalization will also be proved below in Section 12.

3.11. Exercise. The logarithmic spiral in $\mathbf{R}^2$ has a parametric representation $r(\omega) = Ae^{B\omega}$ in polar coordinates where A and B are constants and $A > 0$. It was shown by G. Martin and B. G. Osgood [MAO] that the geodesic segments of k_G, $G = \mathbf{R}^n \setminus \{0\}$, can be obtained as follows. Assume that $x, y \in G$ and that the angle φ between the segments $[0,x]$ and $[0,y]$ satisfies $0 < \varphi < \pi$. Then the triple $0, x, y$ determines a 2-dimensional plane Σ and the geodesic segment of k_G connecting x to y is a logarithmic spiral in Σ with equation

$$r(\omega) = |x| \exp\Big(\frac{\omega}{\varphi} \log \frac{|x|}{|y|}\Big)\ ;\ \ 0 \le \omega \le \varphi\ .$$

Knowing this equation show (by integrating the element of length along this curve) that

$$k_G(x,y) = \sqrt{\varphi^2 + \log^2 \frac{|x|}{|y|}}\ ,\ \ G = \mathbf{R}^n \setminus \{0\}\ , \tag{3.12}$$

holds for all $x, y \in G$. Making use of (3.12) study the set $\{\, z \in G : k_G(e_1, z) = t \,\}$. Note the special case $t = \pi$.

3.13. Exercise. Show that $G = \mathbf{R}^n \setminus \{0\}$ is a uniform domain and that $k_G(x,y) \le A\, j_G(x,y)$ for $x,y \in G$ where

$$A^2 = 1 + \Big(\frac{\pi}{\log 2}\Big)^2 \approx 21.5 \,.$$

[Hint: Let $x,y \in G$ and let $\varphi \in [0,\pi]$ be the angle between x and y. By a property of the bisector of an angle in a triangle, $\sin \frac{1}{2}\varphi \le \frac{|x-y|}{|x|+|y|}$ and hence

$$\begin{aligned}\varphi &\le 2 \arcsin \frac{|x-y|}{|x|+|y|} \le \pi \frac{|x-y|}{|x|+|y|} \\ &\le \frac{\pi}{\log 2} \log\Big(1 + \frac{|x-y|}{|x|+|y|}\Big) \le \frac{\pi}{\log 2}\, j_G(x,y) \;.\end{aligned}$$

By 2.36(2) and (3.12) we obtain the desired inequality.]

3.14. Exercise. Show that $G = \mathbf{B}^2 \setminus [0, e_1)$ is not a uniform domain. [Hint: For $t \in (0, \frac{1}{10})$ let $x_t = (\frac{1}{4}, t)$ and $y_t = (\frac{1}{4}, -t)$, $Y = \{\,(0,y) : y > 0\,\}$. Show that

$$k_G(x_t, y_t) \ge k_G(x_t, Y) = k_{\mathbf{H}^2}(x_t, Y) \longrightarrow \infty$$

when $t \to 0$ (cf. (2.7)), while $j_G(x_t, y_t) = \log 3$ for all $t \in (0, \frac{1}{10})$.]

3.15. Exercise. Let $t \in (0,1)$. Show that

$$D_G\big(x, \log(1+t)\big) \subset B^n\big(x, t\, d(x)\big) \subset D_G\Big(x, \log\frac{1}{1-t}\Big) \;.$$

[Hint: Apply (3.9).]

3.16. Exercise. Suppose that there exists $C \ge 1$ such that for all $x,y \in G$

$$k_G(x,y) \le C \sup\{\, k_{\mathbf{R}^n \setminus \{z\}}(x,y) : z \in \partial G \} \;.$$

Show that G is uniform. [Hint: Apply 3.13 and (2.39).]

3.17. Exercise. Let $f\colon \mathbf{R}^n \to \mathbf{R}^n$ be an L-bilipschitz mapping, that is

$$|x-y|/L \le |f(x) - f(y)| \le L\,|x-y|$$

for all $x,y \in \mathbf{R}^n$, and let $G \subset \mathbf{R}^n$ be uniform. Show that fG is uniform. [Hint: Using the definitions show that

$$k_G(x,y)/L^2 \le k_{fG}\big(f(x), f(y)\big) \le L^2 k_G(x,y) \;.$$

Then deduce from (3.6) that

$$j_G(x,y)/L^2 \le j_{fG}(f(x),f(y)) \le L^2 j_G(x,y)\ .]$$

3.18. Exercise. Let $G = \mathbf{R}^n \setminus \{0\}$ and $f(x) = a^2 x/|x|^2$ for $x \in G$, where $a > 0$. Show that

$$k_G(f(x),f(y)) = k_G(x,y)$$

for $x, y \in G$. [Hint: Apply (3.12).] Show also that

$$j_G(f(x),f(y)) = j_G(x,y)$$

for $x, y \in G$. [Hint: Apply (1.5).] Note that these assertions do not follow from 3.10.

3.19. The symmetric ratio. For distinct points a, b, c, d in $\overline{\mathbf{R}}^n$ define the *symmetric ratio* by

$$s(a,b,c,d) = |\,a,b,d,c\,||\,a,c,d,b\,|\ . \tag{3.20}$$

Then by (1.27)

$$s(a,b,c,d) = \frac{q(a,d)^2\, q(b,c)^2}{q(a,b)\, q(b,d)\, q(a,c)\, q(c,d)}\ , \tag{3.21}$$

which we recognize as the expression studied in 1.32. We recall that s is symmetric, i.e.

$$s(a,b,c,d) = s(a,c,b,d) = s(d,b,c,a) = s(b,a,d,c)\ ,$$

and also $\mathcal{GM}(\overline{\mathbf{R}}^n)$ –invariant in the sense that

$$s_f(a,b,c,d) = s(fa,fb,fc,fd) = s(a,b,c,d) \tag{3.22}$$

for all f in $\mathcal{GM}(\overline{\mathbf{R}}^n)$. Let $D \subset \overline{\mathbf{R}}^n$ be an open set with $\mathrm{card}(\overline{\mathbf{R}}^n \setminus D) \ge 2$ and define

$$s_D(b,c) = \sup\{\ \tfrac{1}{2} s(a,b,c,d) : a, d \in \partial D\ \}\ . \tag{3.23}$$

It follows from (3.21) and (1.15) that

$$s(a,x,y,\infty) = \frac{|x-y|^2}{|a-x||y-a|}\ . \tag{3.24}$$

3.25. The point–pair invariant s_G. We next list some immediate properties of the function $s_G(b,c)$ when $G \subset \overline{\mathbf{R}}^n$.

(1) $s_G(x,y) = s_G(y,x)$,

(2) $s_{fG}(f(x), f(y)) = s_G(x,y)$ for $f \in \mathcal{GM}(\overline{\mathbf{R}}^n)$ and $x,y \in G$,

(3) $G' \subset G$ and $x,y \in G'$ imply $s_{G'}(x,y) \geq s_G(x,y)$,

(4) for fixed $y \in G$, $s_G(x,y) \to 0$ iff $x \to y$ and $s_G(x,y) \to \infty$ iff $x \to \partial G$,

(5) $s_G(x,y) \geq (q(\partial G)\, q(x,y))^2$.

3.26. Lemma. *$s_{\mathbf{B}^n}(b,c) = \operatorname{ch} \rho_{\mathbf{B}^n}(b,c) - 1$ for $b,c \in \mathbf{B}^n$.*

Proof. Because this equality is $\mathcal{GM}(\mathbf{B}^n)$-invariant, we may assume that $b = -re_1 = -c$, $r \in (0,1)$. Then $r = \operatorname{th} \frac{1}{4}\rho(x,y)$ by (2.25). It follows from (3.20) that for $a,d \in S^{n-1}$ we obtain

$$s(a,b,c,d) = \frac{4r^2|a-d|^2}{|a-b||b-d||a-c||c-d|} = \frac{4r^2|a-d|^2}{|a-b||a-c||d-b||d-c|} .$$

It is left as an exercise for the reader to show that

$$\min\{\, |a-b||a-c| : a \in S^{n-1} \,\} = 1 - r^2 ,$$

and similarly for $|d-b||d-c|$. Thus

$$s = \sup\{\, s(a,b,c,d) : a,d \in S^{n-1} \,\} \leq \frac{4r^2 2^2}{(1-r^2)^2} = \left(\frac{4r}{1-r^2}\right)^2 .$$

This upper estimate is in fact attained if $a = -e_1 = -d$. Hence

$$s = 16 \operatorname{sh}^2\left(\tfrac{1}{4}\rho(b,c)\right) \operatorname{ch}^2\left(\tfrac{1}{4}\rho(b,c)\right) = 4 \operatorname{sh}^2\left(\tfrac{1}{2}\rho(b,c)\right) = 2\left(\operatorname{ch}\rho(b,c) - 1\right)$$

and $s_{\mathbf{B}^n}(b,c) = \operatorname{ch}\rho(b,c) - 1$ as desired. ☐

It is clear by (2.22) that Lemma 3.26 holds for the half-space $\mathbf{H}^n$, too. Note also that Lemma 3.26 yields a formula for $\rho(b,c)$ involving the absolute ratio

(3.27) $$\operatorname{ch}\rho(b,c) = 1 + \sup\{\, \tfrac{1}{2}|\,a,b,d,c\,||\,a,c,d,b\,| \;:\; a,d \in S^{n-1} \,\} .$$

Recall that a different formula was given in (2.19). An advantage of (3.27) over (2.19) is that it generalizes to any domain G in $\overline{\mathbf{R}}^n$ with $\operatorname{card}(\overline{\mathbf{R}}^n \setminus G) \geq 2$. For such a domain G define a function ρ_G by

(3.28) $$\operatorname{ch}\rho_G(b,c) = 1 + s_G(b,c) ,$$

when $b,c \in G$.

3.29. Remark. It is an interesting question whether the function ρ_G defined in (3.28) is a metric. We shall discuss this below in Exercise 3.31.

3.30. Exercise. Assume that $a \geq 0$ and define b by $\operatorname{ch} b = 1 + \frac{1}{2}a$. Show that

$$\begin{aligned}\log\left(1+\max\left\{a,\sqrt{a}\,\right\}\right) &\leq b \leq \log\left(1+a+\sqrt{a}\,\right)\\ &\leq 2\log\left(1+\max\left\{a,\sqrt{a}\,\right\}\right)\ .\end{aligned}$$

3.31. Exercise. Let $G = \mathbf{R}^n \setminus \{0\}$ and s_G as defined in (3.23). Explicitly, we see by (3.24) that

$$s_G(x,y) = \frac{|x-y|^2}{2|x||y|}\ ,\quad x,\, y \in G\ .$$

Define ρ_G as in (3.28). Applying 3.30 show that

$$j_G(x,y) \leq 2\,\rho_G(x,y) \leq 4\,j_G(x,y)$$

for $x, y \in G$.

3.32. Exercise. (1) Let $D = \mathbf{R}^n \setminus \{0\}$. Show that

$$1 + 2\,q(x,y) \leq \exp k_D(x,y)$$

for $x, y \in D$.

(2) Let $z \in \mathbf{R}^n$ and $G = \mathbf{R}^n \setminus \{z\}$. Show that

$$q(x,y) \leq \tfrac{1}{2}\,c\,(\exp k_G(x,y) - 1)$$

for all $x, y \in G$, where $c = 1 + \frac{1}{2}|z|\left(|z| + \sqrt{4+|z|^2}\,\right)$. [Hint: Let $D = \mathbf{R}^n \setminus \{0\}$ and $h(x) = x - z$. Then by part (1) and 1.54

$$\begin{aligned}k_G(x,y) &= k_D(x-z,\, y-z) \geq \log(1 + 2\,q(x-z,\, y-z))\\ &\geq \log(1 + 2\,q(x,y)/c)\end{aligned}$$

where c is as above.] *Conclusion:* If G is a proper subdomain of $\mathbf{R}^n$, then (cf. 3.3)

$$\exp k_G(x,y) \geq 1 + 2\,q(x,y)/A$$

where A depends only on $\min\{\,|z| : z \in \partial G\,\}$.

3.33. Exercise. (1) Let $f\colon [0,\infty) \to [0,\infty)$ be increasing with $f(0) = 0$ such that $f(t)/t$ is decreasing on $(0,\infty)$. Show that $f(s+t) \leq f(s) + f(t)$ for $s, t \geq 0$.

(2) Let (X,d) be a metric space and let f be as in part (1). Show that $(X,f\circ d)$ is a metric space, too.

(3) Let (X,d) be a metric space and let $d_1(x,y)=\max\{\,d(x,y),\,d(x,y)^\alpha\,\}$, $0\le\alpha\le 1$. Show that (X,d_1) is a metric space, too.

(4) Give an example of a metric space (Y,d) such that d^β does not satisfy the triangle inequality for any $\beta>1$.

3.34. Notes. The quasihyperbolic metric has been developed by F. W. Gehring and his students. Several interesting results can be found in [GP], [GOS], [MA], [MAO]. Since 1978, when uniform domains were introduced by O. Martio and J. Sarvas [MS2], they have found many interesting applications, e.g. in P. Jones' works [J1], [J2] on extension operators of function spaces. An exposition of these results occurs in [G8], with several equivalent definitions of plane uniform domains. The above variant of the definition of a uniform domain is suggested by [GOS] and [VU10].

4. Some covering problems

In this section we shall consider some geometric problems related to the hyperbolic or quasihyperbolic metric. A typical question, which we are going to answer, is the following. Let X be a compact set in $\mathbf{B}^n$ and let $\mathcal{F}$ be a covering of X by hyperbolic balls with fixed radii. The problem is to extract a subcovering $\mathcal{F}_1$ of $\mathcal{F}$ with $X\subset\bigcup\mathcal{F}_1$ and to give a quantitative upper bound for $\operatorname{card}\mathcal{F}_1$ in terms of the parameters of the problem.

4.1. (a,b,s)–admissible families. Let G be a proper subdomain of $\mathbf{R}^n$, $a,b\in G$, and $s\in(0,1)$. A family $\mathcal{F}=\{\,B^n(x_i,r_i):i=1,\ldots,p\}$ of balls in G is said to be (a,b,s)*–admissible* if the following two conditions are satisfied:

$$(4.2)\qquad \begin{cases} (1) & a\in\overline{B}^n(x_1,sr_1)\,,\ \ b\in\overline{B}^n(x_p,sr_p)\,,\\ (2) & \overline{B}^n(x_j,sr_j)\cap\overline{B}^n(x_{j+1},sr_{j+1})\neq\emptyset\,,\ \ j=1,\ldots,p-1\,. \end{cases}$$

We shall show that the smallest possible number of balls in an (a,b,s)–admissible family is roughly proportional to $k_G(a,b)$, with a constant of proportionality $c(s)$. The case $G=\mathbf{H}^n$ will be studied first. To this end note that by (2.6)

$$(4.3)\qquad \rho\big(\overline{B}^n(x,tx_n)\big)=\log\frac{1+t}{1-t}\,,\quad t\in(0,1)$$

for $x = (x_1, \dots, x_n) \in \mathbf{H}^n$.

4.4. Lemma. *Let* $a, b \in \mathbf{H}^n$ *and* $s \in (0,1)$.

(1) *There is an* (a,b,s) *-admissible family containing at most* $1 + \rho(a,b) / \log \frac{1+s}{1-s}$ *balls.*

(2) *Every* (a,b,s) *-admissible family contains at least* $\rho(a,b)/\log \frac{1+s}{1-s}$ *balls.*

Proof. (1) Choose an integer $p \geq 1$ such that

$$(4.5) \qquad (p-1)\log\frac{1+s}{1-s} \leq \rho(a,b) < p\log\frac{1+s}{1-s} \ .$$

Select points $y_0 = a$, $y_j \in J[a,b]$ such that $\rho(y_0, y_j) = j\log\frac{1+s}{1-s}$, $j = 1, \dots, p-1$, and set $y_p = b$. Let $B^n(x_j, \, sx_{jn})$ be chosen so that $S^{n-1}(x_j, \, sx_{jn})$ is perpendicular to $J[y_{j-1}, y_j]$ and $y_{j-1}, y_j \in S^{n-1}(x_j, sx_{jn})$, $j = 1, \dots, p$. (In other words, $B^n(x_j, \, sx_{jn}) = D(z_j, M)$, where $2M = \log\frac{1+s}{1-s}$ and $z_j \in J[y_{j-1}, y_j]$ and $\rho(z_j, y_{j-1}) = \rho(z_j, y_j) = M$, $1 \leq j \leq p-1$.) Here x_{jn} is the nth coordinate of x_j . In view of (4.3) and (4.5) the family $\{ B^n(x_j, x_{jn}) : j = 1, \dots, p \}$ is the desired (a,b,s)-admissible family.

(2) Suppose that $\{ B^n(z_j, r_j) \, : \, j = 1, \dots, m \}$ is (a,b,s)-admissible. By (4.3) we get

$$\rho(a,b) \leq \sum_{j=1}^{m} \rho\big(\overline{B}^n(z_j, \, sr_j)\big) \leq \sum_{j=1}^{m} \rho\big(\overline{B}^n(z_j, \, sz_{jn})\big) = m\log\frac{1+s}{1-s} \ ,$$

from which the desired lower bound follows. ❐

Before formulating an analogue of Lemma 4.4 for $\mathbf{B}^n$ we make a few observations about hyperbolic balls. By (2.23) $D(x,M) = B^n(y,r)$ with

$$\frac{r}{1-|y|} = \frac{(1+|x|)t}{1+|x|t^2} \in \left[\operatorname{th}\tfrac{1}{2}M, \, \operatorname{th} M \right] \ ; \ \ t = \operatorname{th}\tfrac{1}{2}M \ ,$$

for all $x \in \mathbf{B}^n$ (see Exercise 2.31) and hence

$$(4.6) \qquad B^n\big(y, \, (\operatorname{th}\tfrac{1}{2}M)(1-|y|)\,\big) \subset D(x,M) \subset B^n\big(y, \, (\operatorname{th} M)(1-|y|)\,\big) \ .$$

It follows from (4.6) that

$$(4.7) \qquad \log\frac{1+s}{1-s} \leq \rho\big(B^n(z, \, s(1-|z|)\,)\big) \leq 2\log\frac{1+s}{1-s} \ .$$

4.8. Lemma. *Let* $a, b \in \mathbf{B}^n$ *and* $s \in (0,1)$.

(1) *There is an* (a,b,s) *-admissible family containing at most* $1+\rho(a,b)/\log\frac{1+s}{1-s}$ *balls.*

(2) *Every* (a,b,s) *-admissible family contains at least* $\rho(a,b)/(2\log\frac{1+s}{1-s})$ *balls.*

Proof. As in the proof of Lemma 4.4 we cover $J[a,b]$ by $\{\overline{D}(x_j,M)\ :\ j=1,\ldots,p\}$, $p \le 1+\frac{1}{2M}\rho(a,b)$ where M is chosen so that $\overline{D}(x_j,M) \subset B^n(y_j,\, s(1-|y_j|))$. By (4.6) we may choose $M=\frac{1}{2}\log\frac{1+s}{1-s}$. The proof of (1) follows now as in Lemma 4.4(1). The proof of (2) is similar to the proof of part (2) of Lemma 4.4 except that here we use the two-sided inequality (4.7) instead of the equality (4.3). □

In the next lemma we prove a counterpart of Lemma 4.8 for an arbitrary domain.

4.9. Lemma. *Let* G *be a proper subdomain of* $\mathbf{R}^n$, $a, b \in G$ *and* $0 < s < 1$.

(1) *There is an* (a,b,s) *-admissible family containing at most* $1+k_G(a,b)/d_1(s)$ *balls,* $d_1(s)=2\log(1+s)$.

(2) *Every* (a,b,s) *-admissible family contains at least* $k_G(a,b)/d_2(s)$ *balls,* $d_2(s)=2\log\frac{1}{1-s}$.

Proof. (1) Fix a quasihyperbolic geodesic segment $J_G[a,b]$. Choose points $z_j \in J_G[a,b]$, $j=1,\ldots,p$ with $k_G(a,z_1)=M$, $k_G(z_j,z_{j+1})=2M$, $j=1,\ldots,p-2$, $z_p=b$, $k_G(z_{p-1},z_p)<2M$, where $M>0$ will be chosen soon and $2M(p-1)\le k_G(a,b)$. We wish to choose M such that $D_G(z_j,M)\subset B^n(z_j,\, sd(z_j))$, $j=1,\ldots,p$. In view of Exercise 3.15 it suffices to choose M such that $\log(1+s)=M$. It is clear that the family $\{B^n(z_j,d(z_j))\ :\ j=1,\ldots,p\}$ is (a,b,s)-admissible and that $p\le 1+k_G(a,b)/d_1(s)$, $d_1=2\log(1+s)$.

(2) It follows from Exercise 3.15 that for all $y\in G$

$$k_G\big(B^n(y,sd(y))\big)\le 2\log\frac{1}{1-s}\ .$$

The proof follows from this inequality exactly in the same way as in part (2) of Lemma 4.4. □

We shall next give an immediate application of Lemma 4.9 to positive functions satisfying the Harnack inequality.

4.10. Definition. Let G be a proper subdomain of $\mathbf{R}^n$ and let $u\colon G \to \mathbf{R}_+ \cup \{0\}$ be continuous. We say that u satisfies the *Harnack inequality* in G if there exist numbers $s \in (0,1)$ and $C_s \geq 1$ such that

$$\max_{B_x} u(z) \leq C_s \min_{B_x} u(z) \tag{4.11}$$

holds true whenever $B^n(x,r) \subset G$ and $B_x = \overline{B}^n(x,sr)$.

The above definition does not require smoothness or any other regularity properties beyond continuity of u. It is well known that non-negative harmonic functions satisfy (4.11) [GT, p. 16].

4.12. Lemma. *Let $u\colon G \to \mathbf{R}_+ \cup \{0\}$ satisfy the Harnack inequality in G. Then*

$$u(x) \leq C_s^{1+t} u(y) \;, \quad t = \frac{k_G(x,y)}{d_1(s)}$$

for $x, y \in G$ where $d_1(s) = 2\log(1+s)$. If $G = \mathbf{H}^n$ or $G = \mathbf{B}^n$, then we can replace t by $\rho(x,y)/\log\frac{1+s}{1-s}$.

Proof. Fix $x, y \in G$ and an (x,y,s)-admissible family $\{B^n(x_i,r_i) : i = 1,\ldots,p\}$ with $p \leq 1 + k_G(x,y)/d_1(s)$ (see 4.8). Let $z_j \in \overline{B}_j \cap \overline{B}_{j+1}$, $B_j = B^n(x_j, sr_j)$, $j = 1,\ldots,p-1$. By (4.11) we get the desired inequality

$$u(x) \leq C_s u(z_1) \leq C_s^2 u(z_2) \leq \cdots \leq C_s^{p-1} u(z_{p-1}) \leq C_s^p u(y) \;. \quad \square$$

Let u be as in Lemma 4.12. It should be noticed that, by virtue of 4.12, either u vanishes identically or u is strictly positive in G.

4.13. Corollary. *Let $f\colon (G, k_G) \to (\mathbf{R}, |\;|)$ be uniformly continuous as a mapping between metric spaces. Then there is a number α such that*

$$|f(x) - f(y)| \leq 1 + \alpha\, k_G(x,y)$$

for all $x, y \in G$.

Proof. Because f is uniformly continuous, there exists a number t_0 such that $|f(x) - f(y)| \leq 1$ for $x, y \in G$, $k_G(x,y) \leq t_0$. If $k_G(x,y) > t_0$ we can exploit the method of the proof of 4.9 to show that $|f(x) - f(y)| \leq 1 + k_G(x,y)/t_0$. Hence we may choose $\alpha = 1/t_0$. The details are left as an easy exercise for the reader. $\square$

The hyperbolic volume of a (Lebesgue) measurable set E in $\mathbf{B}^n$ is defined by

$$m_h(E) = \int_E \frac{2^n dm(x)}{(1-|x|^2)^n} \tag{4.14}$$

(cf. Section 2). Let ω_{n-1} be the $(n-1)$-dimensional area of S^{n-1}. Integration in polar coordinates yields

$$m_h(B^n(s)) = 2^n \omega_{n-1} \int_0^s \frac{t^{n-1}}{(1-t^2)^n} dt < \frac{2^n \omega_{n-1}}{n-1} \Big(\frac{s}{1-s} \Big)^{n-1} . \tag{4.15}$$

The last inequality holds because

$$\begin{aligned} \int_0^s \frac{t^{n-1}dt}{(1-t^2)^n} &\le s^{n-1} \int_0^s \frac{dt}{(1+t)^n(1-t)^n} \\ &< s^{n-1} \int_0^s \frac{dt}{(1-t)^n} = s^{n-1} \Big/_{\!0}^{\!s} \frac{(1-t)^{1-n}}{n-1} . \end{aligned}$$

Since $t^{n-1} \ge 2^{2-n} t$ for $t \in (\frac{1}{2}, 1)$ we obtain

$$m_h(B^n(s)) > 2^n \omega_{n-1} 2^{2-n} \int_{1/2}^s \frac{tdt}{(1-t^2)^n} > \frac{2^{2(1-n)}\omega_{n-1}}{n-1} (1-s)^{1-n} - \frac{1}{n-1} \tag{4.16}$$

for $s \in (\frac{1}{2}, 1)$. Finally, for $x \in \mathbf{B}^n$ and $M > 0$, by the invariance of m_h under the action of $\mathcal{GM}(\mathbf{B}^n)$ and by (2.24) and (4.15) we get

$$\begin{aligned} & m_h(D(x,M)) = m_h(D(0,M)) = m_h(B^n(\operatorname{th} \tfrac{1}{2}M)) \\ & \le \frac{2^n \omega_{n-1}}{n-1} \Big(\frac{\operatorname{th}\frac{1}{2}M}{1-\operatorname{th}\frac{1}{2}M} \Big)^{n-1} < \frac{2^n \omega_{n-1}}{n-1} e^{M(n-1)} \operatorname{th}^{n-1}(\tfrac{1}{2}M) . \end{aligned} \tag{4.17}$$

For what follows we shall need a lemma about coverings by families of euclidean balls [LA, p. 197, Lemma 3.2]. We shall give such a lemma here with a slightly more general formulation. Covering theorems of this sort are very useful in analysis. For a related result see [GU, Theorem 1.1].

4.18. Lemma. *Let (X,d) be any one of the metric spaces $(\mathbf{R}^n, |\ |)$, $(\mathbf{B}^n, \rho_{\mathbf{B}^n})$, or $(\mathbf{H}^n, \rho_{\mathbf{H}^n})$, and let $B_X(z,r) = \{ y \in X : d(y,z) < r \}$. Let A be a non-empty subset of X, $\mathcal{F} = \{ B_X(z, r(z)) : z \in A \}$ and suppose $\sup\{ r(x) : x \in A \} < \infty$. There exists a number $c(n)$ depending only on n and a countable subfamily $\mathcal{F}_1 \subset \mathcal{F}$ such that (1) $A \subset \bigcup \mathcal{F}_1$ and (2) each $x \in A$ belongs to at most $c(n)$ elements of $\mathcal{F}_1$.*

Let $A \subset X$, $A \neq \emptyset$, where (X,d) is a metric space, and for $t > 0$ set

$$p_X(A,t) = \inf\left\{ k \,:\, A \subset \bigcup_{j=1}^{k} B_X(x_j,t)\,,\ x_j \in A \right\} . \tag{4.19}$$

Because the space X is usually specified by the context, we write $p(A,t) = p_X(A,t)$. Note that if $A \subset X$ is non–empty and compact then $p_X(A,t) < \infty$ and $\{ B_X(x_j,t) : j = 1,\ldots,p_X(A,t) \}$, $x_j \in A$, is a covering of A.

4.20. Lemma. *Let (X,d) be any one of the metric spaces in 4.18, let $m_X = m$ for $X = \mathbf{R}^n$, $m_X = m_h$ if $X = \mathbf{B}^n$ or $X = \mathbf{H}^n$, and let $A \subset X$, $A \neq \emptyset$, be compact. There is a number $d_1 = 1/m_X(B_X(y,t))$ depending only on n and t such that*

$$d_1 m_X(A) \le p(A,t) \le c(n) d_1 m_X\Big(\bigcup_{z\in A} B_X(z,t)\Big)$$

where $c(n)$ is as in 4.18.

The simple proof of this lemma is based on a standard volume–comparison argument and on Lemma 4.18, and left as an easy exercise for the reader.

4.21. Exercise. Show that $m_h\big(\bigcup_{|x|\le r} D(x,M)\big) \le d_2(n,M)(1-r)^{1-n}$, where m_h is the hyperbolic measure of $(\mathbf{B}^n, \rho_{\mathbf{B}^n})$. [Hint: $\bigcup_{|x|\le r} D(x,M) = B^n(R)$ where $\frac{1+R}{1-R} = e^M \frac{1+r}{1-r}$ (see (2.18)). Hence $1 - R \ge (1-r)e^{-M}$, and one may apply (4.15).]

4.22. Exercise. Apply 4.20 and 4.21 to show that for r near 1

$$d_3(1-r)^{1-n} \le p(B^n(r),M) \le c(n)\, d_1\, d_2(1-r)^{1-n}$$

where d_3 depends only on n and M. [Hint: Apply also (4.16).]

4.23. Exercise. For $\varphi \in (0,\frac{1}{2}\pi)$ let $C(\varphi) = \{ z \in \mathbf{R}^n : z \cdot e_n = |z|\cos\varphi \}$ and $A_t^{\varphi} = C(\varphi) \cap \big(B^n(1) \setminus \overline{B}^n(\frac{1}{t})\big)$, $t > 1$. Show that

$$\rho_{\mathbf{H}^n}(A_t^{\varphi}) \le \log\left[\frac{(1+u)^2}{4\cos^2\varphi}\Big(\sqrt{t} + \frac{1}{\sqrt{t}}\Big)^2\right] ,$$

where $u^2 = \sin^2\varphi + (\frac{t-1}{t+1})^2\cos^2\varphi$. [Hint: Consider the smallest euclidean ball B containing A_t^{φ}, and find an upper bound for $\rho(B)$.]

4.24. Remark. For $n = 2$, the hyperbolic area of $D(0,r)$ is $4\pi\,\mathrm{sh}^2(\frac{1}{2}r)$ ([BE, p. 132, Theorem 7.22]). Note that for $r \to 0$ this is approximately πr^2, the euclidean area of $B^2(r)$.

4.25. Exercise. In any ball $B^n(x,r)$ in $\mathbf{R}^n$ one can define a hyperbolic metric ρ_r by making use of a formula similar to (2.20). Generalizing (2.19), we have the equality $\rho_r(x,\, x+ae_1) = \log\frac{1+a/r}{1-a/r}$ for $0 \le a < r$. Assume that $z \in \mathbf{B}^n$, $M > 0$, and ρ is the hyperbolic metric of $\mathbf{B}^n$, and let $\widetilde{\rho}$ be the hyperbolic metric of $D(z,M)$. Let $a \in D(z,M)$ with $\rho\big(a,\, \partial D(z,M)\big) \ge b > 0$. Find an upper bound for $\widetilde{\rho}(z,a)$ in terms of $\rho(z,a)$ and b. [Hint: By the invariance of $\widetilde{\rho}$ and ρ we may assume that $z = 0$, whence $D(z,M) = B^n(\operatorname{th}\frac{1}{2}M)$.]

4.26. Exercise. Let G be a proper subdomain of $\mathbf{R}^n$ and F a connected subset of G with $d(F,\partial G) > 0$. Applying the covering lemma 4.18 show that

$$k_G(F) \le c\Big(n, \frac{d(F)}{d(F,\partial G)}\Big) < \infty\,.$$

[Hint: See [VU5, 2.18].]

4.27. Notes. Chains of balls similar to those in Lemmas 4.4 and 4.8, but often without a quantitative upper bound for the number of balls, are recurrent in analysis. With slightly different constants, 4.4 and 4.8 were given in [VU5], [VU6]. For 4.9 see [HVU]. Some formulae for the hyperbolic volume or area are given in [BE], [A5]. Instead of balls one could use cubes in Lemma 4.18, see [GU, Theorem 1.1].

Chapter II
MODULUS AND CAPACITY

For non–empty subsets E and F of $\overline{\mathbf{R}}^n$ let Δ_{EF} be the family of all curves joining E and F in $\overline{\mathbf{R}}^n$. For fixed F the modulus $\mathsf{M}(\Delta_{EF})$ of Δ_{EF} is an outer measure defined for compact subsets E of $\overline{\mathbf{R}}^n \setminus F$. The real number $\mathsf{M}(\Delta_{EF})$ gives quantitative information about the structure of the sets E and F as well as their position relative to each other. Roughly speaking $\mathsf{M}(\Delta_{EF})$ is small if E and F are far apart or if one of the sets E, F is "thin", while the modulus is large in the opposite case. If E and F are non–degenerate continua in $\mathbf{R}^n$, then $\mathsf{M}(\Delta_{EF})$ and $\min\{d(E), d(F)\}/d(E,F)$ are simultaneously small or large. Because of its conformal invariance, the modulus will be a most valuable tool in our subsequent studies in Chapter III.

We shall exploit the conformal invariance of the modulus and introduce in a subdomain G of $\overline{\mathbf{R}}^n$ two conformal invariants $\lambda_G(x,y)$ and $\mu_G(x,y)$, $x,y \in G$, which describe the position of x and y with respect to each other and the boundary of G. One may think of $\mu_G(x,y)$ as a conformally invariant "intrinsic metric" of G while $\lambda_G(x,y)$ is in a sense its dual quantity. The importance of μ_G and λ_G for Chapter III is based largely on the explicit estimates proved in this chapter as well as on the fact that μ_G and λ_G transform in a natural way under quasiconformal and quasiregular mappings.

5. The modulus of a curve family

For the sake of easy reference and for the reader's convenience we shall give in this section the basic properties of the modulus of a curve family. The proofs of several well-known results are omitted. For the proofs of these results and for more details the reader is referred to original sources which we shall quote at the end of this section. Most of the material in Section 5 is based on Chapter I of Väisälä's book [V7].

A *path* in $\mathbf{R}^n$ ($\overline{\mathbf{R}}^n$) is a continuous mapping $\gamma\colon \Delta \to \mathbf{R}^n$ (resp. $\overline{\mathbf{R}}^n$) where $\Delta \subset \mathbf{R}$ is an interval. If $\Delta' \subset \Delta$ is an interval, we call $\gamma|\Delta'$ a subpath of γ. The path γ is called *closed* (*open*) if Δ is *closed* (resp. *open*). (Note that according to this definition, e.g. the path $\gamma\colon [0,1] \to \mathbf{R}^n$ is closed and that it is not required that $\gamma(0) = \gamma(1)$.) The *locus* (or *trace*) of a path γ is the set $\gamma\Delta$. The locus is also denoted by $|\gamma|$ or simply by γ if there is no danger of confusion. We use the word *curve* as a synonym for *path*. The *length* $\ell(\gamma)$ of a curve $\gamma\colon \Delta \to \mathbf{R}^n$ is defined in the usual way, with the help of polygonal approximations and a passage to the limit (see [V7, pp. 1–8]). The path $\gamma\colon \Delta \to \mathbf{R}^n$ is called *rectifiable* if $\ell(\gamma) < \infty$ and *locally rectifiable* if each closed subpath of γ is rectifiable. If $\gamma\colon [a,b] \to \mathbf{R}^n$ is a rectifiable path, then γ has a parametrization by means of arc length, also called the *normal representation* of γ. The normal representation of γ is denoted by $\gamma^0\colon [0,\ell(\gamma)] \to \mathbf{R}^n$. Making use of the normal representation one defines the line integral over a rectifiable curve γ. In a natural way one then extends the definition to locally rectifiable curves (for a thorough discussion see [V7, pp. 1–15]).

Let Γ be a family of curves in $\mathbf{R}^n$. By $\mathcal{F}(\Gamma)$ we denote the family of *admissible* functions, i.e. non-negative Borel-measurable functions $\rho\colon \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ such that

$$\int_\gamma \rho\, ds \geq 1$$

for each locally rectifiable curve γ in Γ. For $p \geq 1$ the *p-modulus* of Γ is defined by

$$\mathsf{M}_p(\Gamma) = \inf_{\rho \in \mathcal{F}(\Gamma)} \int_{\mathbf{R}^n} \rho^p\, dm\,, \tag{5.1}$$

where m stands for the n-dimensional Lebesgue measure. If $\mathcal{F}(\Gamma) = \emptyset$, we set $\mathsf{M}_p(\Gamma) = \infty$. The case $\mathcal{F}(\Gamma) = \emptyset$ occurs only if there is a constant path in Γ because otherwise the constant function ∞ is in $\mathcal{F}(\Gamma)$. Usually $p = n$ and we denote $\mathsf{M}_n(\Gamma)$ also by $\mathsf{M}(\Gamma)$ and call it the *modulus* of Γ. If $\mathsf{M}(\Gamma) > 0$, the number $\mathsf{M}(\Gamma)^{1/(1-n)}$ is called the *extremal length* of Γ. We take the extremal length to be ∞ if $\mathsf{M}(\Gamma) = 0$.

5.2. Lemma. *The p-modulus M_p is an outer measure in the space of all curve families in $\mathbf{R}^n$. That is,*

(1) $\mathsf{M}_p(\emptyset) = 0$,

(2) $\Gamma_1 \subset \Gamma_2$ *implies* $\mathsf{M}_p(\Gamma_1) \leq \mathsf{M}_p(\Gamma_2)$,

(3) $\mathsf{M}_p\Big(\bigcup_{i=1}^{\infty} \Gamma_i\Big) \leq \sum_{i=1}^{\infty} \mathsf{M}_p(\Gamma_i)$.

Let Γ_1 and Γ_2 be curve families in $\mathbf{R}^n$. We say that Γ_2 is *minorized* by Γ_1 and write $\Gamma_2 > \Gamma_1$ if every $\gamma \in \Gamma_2$ has a subcurve belonging to Γ_1.

5.3. Lemma. $\Gamma_1 < \Gamma_2$ *implies* $\mathsf{M}_p(\Gamma_1) \geq \mathsf{M}_p(\Gamma_2)$.

The curve families $\Gamma_1, \Gamma_2, \ldots$ are called *separate* if there exist disjoint Borel sets E_i in $\mathbf{R}^n$ such that if $\gamma \in \Gamma_i$ is locally rectifiable then $\int_\gamma \chi_i ds = 0$ where χ_i is the characteristic function of $\mathbf{R}^n \setminus E_i$.

5.4. Lemma. *If* $\Gamma_1, \Gamma_2, \ldots$ *are separate and if* $\Gamma < \Gamma_i$ *for all* i, *then*

$$\mathsf{M}_p(\Gamma) \geq \sum \mathsf{M}_p(\Gamma_i) .$$

5.5. Lemma. *Let* G *be a Borel set in* $\mathbf{R}^n$ *and* $\Gamma = \{ \gamma : \gamma$ *is a curve in* G *with* $\ell(\gamma) \geq r \}$. *If* $r > 0$ *then*

$$\mathsf{M}_p(\Gamma) \leq m(G) r^{-p} .$$

Proof. Because $\rho = \frac{1}{r}\chi_G \in \mathcal{F}(\Gamma)$ the proof follows from (5.1). ❐

5.6. Corollary. *If* Γ *is the family of non-constant curves in a Borel set* $G \subset \mathbf{R}^n$ *with* $m(G) = 0$, *then* $\mathsf{M}_p(\Gamma) = 0$.

Proof. If $\Gamma_j = \{ \gamma \in \Gamma : \ell(\gamma) \geq \frac{1}{j} \}$, $j = 1, 2, \ldots$, then $\Gamma = \bigcup \Gamma_j$ and the proof follows from 5.2(3) and 5.5. ❐

Curve families with zero p-modulus are sometimes called *p-exceptional.* We next give a general criterion for a curve family to be p-exceptional, which is a generalization of 5.6.

5.7. Lemma. *A curve family* Γ *is* p*-exceptional if and only if there exists an admissible function* $\rho \in \mathcal{F}(\Gamma)$ *such that*

$$\int_{\mathbf{R}^n} \rho^p \, dm < \infty \quad \text{and} \quad \int_\gamma \rho \, ds = \infty$$

for every locally rectifiable $\gamma \in \Gamma$.

Proof. If ρ satisfies the above conditions, then $k^{-1}\rho \in \mathcal{F}(\Gamma)$ for every $k = 1, 2, \ldots$ and thus

$$\mathsf{M}_p(\Gamma) \le k^{-p} \int_{\mathbf{R}^n} \rho^p \, dm \longrightarrow 0$$

as $k \to \infty$. Hence Γ is p-exceptional. Conversely, let $\mathsf{M}_p(\Gamma) = 0$ and choose a sequence $\rho_k \in \mathcal{F}(\Gamma)$ such that $\int_{\mathbf{R}^n} \rho_k^p \, dm < 4^{-k}$, $k = 1, 2, \ldots$. Writing

$$\rho(x) = \Big(\sum_{k=1}^{\infty} 2^k \rho_k(x) \Big)^{1/p}$$

we infer that $\int_{\mathbf{R}^n} \rho^p \, dm < \infty$. On the other hand

$$\int_\gamma \rho \, ds \ge \int_\gamma 2^{k/p} \rho_k \, ds \ge 2^{k/p}$$

for all $k = 1, 2, \ldots$ i.e. $\int_\gamma \rho \, ds = \infty$ for each locally rectifiable curve γ in Γ. ❐

5.8. Corollary. *If* Γ *is a curve family in* $\mathbf{R}^n$ *and* $\Gamma_r = \{ \gamma \in \Gamma : \ell(\gamma) < \infty \}$, *then* $\mathsf{M}(\Gamma) = \mathsf{M}(\Gamma_r)$.

Proof. Set $\rho(x) = 1$ for $|x| < 2$ and $\rho(x) = 1/(|x| \log |x|)$ for $|x| \ge 2$. By direct computation

$$\int_{\mathbf{R}^n} \rho^n \, dm = 2^n \Omega_n + \frac{\omega_{n-1}}{(n-1)(\log 2)^{n-1}} < \infty ,$$

where Ω_n is the n-dimensional volume of $\mathbf{B}^n$ and ω_{n-1} is the $(n-1)$-dimensional area of S^{n-1}. Let $\Gamma_\infty = \{ \gamma \in \Gamma : \ell(\gamma) = \infty \}$. In view of 5.7 it suffices to show that $\int_\gamma \rho \, ds = \infty$ for all $\gamma \in \Gamma_\infty$. If γ is bounded, then $\rho(x) \ge a > 0$ on $|\gamma|$ and it is clear that $\int_\gamma \rho \, ds = \infty$. If $\gamma \in \Gamma_\infty$ is unbounded we choose $x \in |\gamma| \setminus B^n(2)$. It follows that

$$\int_\gamma \rho \, ds \ge \int_{|x|}^{\infty} \frac{dr}{r \log r} = \infty$$

as desired. ❐

For $E, F, G \subset \overline{\mathbf{R}}^n$ we denote by $\Delta(E, F; G)$ the family of all closed non-constant curves joining E and F in G. More precisely, a non-constant path $\gamma \colon [a, b] \to \overline{\mathbf{R}}^n$ belongs to $\Delta(E, F; G)$ iff (1) one of the end points $\gamma(a), \gamma(b)$ belongs to E and the other to F, and (2) $\gamma(t) \in G$ for $a < t < b$.

5.9. Remark. If $G = \mathbf{R}^n$ or $\overline{\mathbf{R}}^n$ we often denote $\Delta(E,F;G)$ by $\Delta(E,F)$. Curve families of this form are the most important for what follows. The following subadditivity property is useful. If $E = \bigcup_{j=1}^{\infty} E_j$ and $c_E(F) = \mathsf{M}_p(\Delta(E,F)) = c_F(E)$, then $c_F(E) \le \sum c_F(E_j)$, see 5.2(3). More precisely if $G \subset \overline{\mathbf{R}}^n$ is a domain and $F \subset G$ is fixed, then $c_F^G(E) = \mathsf{M}_p(\Delta(E,F;G))$ is an outer measure defined for $E \subset G$. In a sense which will be made precise later on, $c_E(F)$ describes the mutual size and location of E and F. Assume now that D is an open set in $\overline{\mathbf{R}}^n$ and that $F \subset D$. It follows from 5.2(2) that

$$\mathsf{M}_p(\Delta(F,\partial D; D\setminus F)) \le \mathsf{M}_p(\Delta(F,\partial D;D)) \le \mathsf{M}_p(\Delta(F,\partial D)) .$$

On the other hand, because $\Delta(F,\partial D;D) < \Delta(F,\partial D)$ and $\Delta(F,\partial D; D\setminus F) < \Delta(F,\partial D;D)$, 5.3 yields

$$\mathsf{M}_p(\Delta(F,\partial D)) = \mathsf{M}_p(\Delta(F,\partial D;D)) = \mathsf{M}_p(\Delta(F,\partial D;D\setminus F)) . \tag{5.10}$$

As the relatively complicated definition (5.1) of the p-modulus suggests, it is usually a very difficult task to find $\mathsf{M}_p(\Gamma)$ when Γ is given. In fact, the real number $\mathsf{M}_p(\Gamma)$ is known for very few curve families. If Γ has a simple structure, then one can sometimes compute $\mathsf{M}_p(\Gamma)$ in two steps. First, applying Hölder's inequality and Fubini's theorem one proves a lower bound for $\int_{\mathbf{R}^n} \rho^n dm$ when ρ is an admissible function. Second, one shows that this lower bound is attained by some particular admissible function ρ_1. Making use of this method one can compute the modulus of a cylinder and of a spherical ring (for details see Väisälä [V7, pp. 20–23]).

5.11. The cylinder. Let $E \subset \{ x \in \mathbf{R}^n : x_n = 0 \}$ be a Borel set, $h > 0$, $F = E + he_n$ and denote

$$G = \{ x \in \mathbf{R}^n : (x_1, \ldots, x_{n-1}, 0) \in E,\ 0 < x_n < h \} .$$

Then G is a cylinder with bases E and F and, as shown in [V7]

$$\mathsf{M}_p(\Delta(E,F;G)) = m_{n-1}(E)\, h^{1-p} = m(G)\, h^{-p} .$$

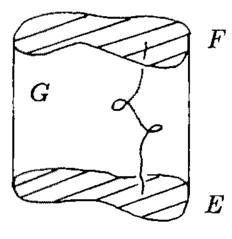


Diagram 5.1.

5.12. The spherical ring. Let $0 < a < b$, $D = \overline{B}^n(b) \setminus B^n(a)$ and let Y be a Borel set in S^{n-1}. Let $\gamma_y = \{ ty : a \le t \le b \}$, $y \in Y$. Then [V7]

$$\mathsf{M}(\Gamma) = m_{n-1}(Y)\Big(\log\frac{b}{a}\Big)^{1-n}; \quad \Gamma = \{ \gamma_y : y \in Y \}, \tag{5.13}$$

$$\mathsf{M}\big(\Delta(S^{n-1}(b), S^{n-1}(a); D)\big) = \omega_{n-1}\Big(\log\frac{b}{a}\Big)^{1-n}. \tag{5.14}$$

By (5.10) the formula (5.14) holds also if D is replaced by $\mathbf{R}^n$. Letting $a \to 0$ we see by 5.3 that

$$\mathsf{M}\big(\Delta(S^{n-1}(b), \{0\})\big) = 0. \tag{5.15}$$

It follows from (5.15), 5.2, and 5.3 (see the proof of 5.6 or [V7, p. 23]) that the family of all non-constant curves γ passing through a prescribed point $x_0 \in \mathbf{R}^n$ is n-exceptional.

5.16. Remark. The inequality of Lemma 5.5 is sharp: if G is the cylinder in 5.11 with bases E and F then 5.5 holds as an equality. However, this is not usually the case. Applying (5.14) we shall now give an example in which Lemma 5.5 gives a very crude estimate. Let $E = S^{n-1}$, $F = S^{n-1}(3)$, $G_t = \big(B^n(4) \setminus S^{n-1}(2)\big) \cup B^n(2e_1, t)$, and $\Gamma_t = \Delta(E, F; G_t)$, $t \in (0, \frac{1}{2})$. In this particular case, Lemma 5.5 yields for $\mathsf{M}(\Gamma_t)$ an upper bound independent of t. Let $\Delta_t = \Delta(S^{n-1}(2e_1, t), S^{n-1}(2e_1, 1); G_t)$. Because $\Delta_t < \Gamma_t$, we get by (5.14)

$$\mathsf{M}(\Gamma_t) \le \omega_{n-1}\Big(\log\frac{1}{t}\Big)^{1-n}$$

i.e. $\mathsf{M}(\Gamma_t) \to 0$ as $t \to 0$.

In conclusion, keeping E and F fixed and letting the domain G_t vary so that $E, F \subset G_t$ and $m(G_t)$ is constant, one can make $\mathsf{M}\big(\Delta(E, F; G_t)\big)$ arbitrarily small, while this fact is not reflected in the form of the upper bound 5.5.

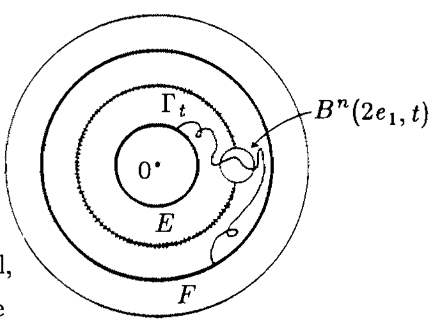


Diagram 5.2.

The most valuable property of the n-modulus is invariance under conformal mappings, which is the content of the next lemma.

We first extend the definition (5.1) to curve families in $\overline{\mathbf{R}}^n$ when $p = n$. Let Γ be a curve family in $\overline{\mathbf{R}}^n$. If Γ contains a constant curve, we set $\mathsf{M}(\Gamma) = \infty$.

Denote $\Gamma_\infty = \{\gamma \in \Gamma : \infty \in \gamma\}$. If Γ does not contain a constant curve, we set $\mathsf{M}(\Gamma) = \mathsf{M}(\Gamma \setminus \Gamma_\infty)$. It follows from (5.15), 5.2, and 5.3 that $\mathsf{M}(\Gamma_\infty) = 0$. Hence for curve families in $\mathbf{R}^n$, this extended definition of n-modulus coincides with (5.1). It should be pointed out that we have not included p-modulus, $p \neq n$, in this extended definition (see 5.28).

5.17. Lemma. *Let D and D' be domains in $\overline{\mathbf{R}}^n$ and let $f\colon D \to D'$ be a conformal mapping. Then $\mathsf{M}(f\Gamma) = \mathsf{M}(\Gamma)$ for each curve family Γ in D where $f\Gamma = \{ f \circ \gamma : \gamma \in \Gamma \}$.*

It is easy to see that 5.17 is false for the p-modulus, $p \neq n$, even if f is the stretching $x \mapsto 2x$. Consider e.g. the curve family $\Delta = \Delta(E, F; G)$ where $E = \{ z \in \mathbf{B}^n : z_n = 0 \}$, $F = E + \{h e_n\}$, $h > 0$, and $G = \{ x \in \mathbf{R}^n : x_1^2 + \ldots + x_{n-1}^2 < 1,$ $0 < x_n < 1 \}$. Then by 5.11

$$0 < \mathsf{M}_p(\Delta) = m(G) h^{-p} \neq m(fG)(2h)^{-p} = \mathsf{M}_p(f\Delta) = 2^{n-p}\mathsf{M}_p(\Delta)$$

for $p \neq n$.

An immediate application of the conformal invariance 5.17 is the following counterpart of (5.14) in the hyperbolic and spherical geometries.

5.18. Corollary. *Let $0 < a < b$ and $x \in \mathbf{B}^n$ and*

$$\Delta(a,b) = \Delta\big(\partial D(x,a),\, \partial D(x,b);\, D(x,b) \setminus D(x,a)\big) .$$

Then

$$\text{(1)} \qquad \mathsf{M}(\Delta(a,b)) = \omega_{n-1} L(a,b)^{1-n} ; \quad L(a,b) = \log \frac{\operatorname{th}(b/2)}{\operatorname{th}(a/2)} .$$

If $z \in \overline{\mathbf{R}}^n$, $0 < r < s < 1$, and $\Gamma(r,s) = \Delta(\partial Q(x,r), \partial Q(x,s))$ then

$$\text{(2)} \qquad \mathsf{M}\big(\Gamma(r,s)\big) = \omega_{n-1} \left[\log\left(\frac{s}{r} \sqrt{\frac{1-r^2}{1-s^2}} \right) \right]^{1-n} .$$

Proof. Let $T_x \in M(\mathbf{B}^n)$ be as defined in 1.34. Then $T_x(x) = 0$ and we obtain by (2.25)

$$\text{(5.19)} \qquad T_x D(x,c) = D(0,c) = B^n(\operatorname{th} \tfrac{1}{2}c) , \; c > 0 .$$

Now 5.17 together with (5.19) and (5.14) yields

$$\mathsf{M}(\Delta(a,b)) = \mathsf{M}(T_x\Delta(a,b)) = \omega_{n-1}L(a,b)^{1-n} .$$

For the proof of (2), let $t_x \in M(\overline{\mathbf{R}}^n)$ be a spherical isometry with $t_x(x) = 0$, see (1.46). Then by (1.47) or 1.25(1)

$$t_x Q(x,r) = Q(0,r) = B^n\big(r/\sqrt{1-r^2}\,\big) .$$

The proof of (2) follows from this equality, 5.17, and (5.14). □

Next we shall discuss various symmetry properties of the modulus. If $A \subset \mathbf{R}^n_+$ we denote by A^* the symmetric image $\{(x_1,\ldots,x_{n-1},-x_n) \in \mathbf{R}^n : (x_1,\ldots,x_n) \in A\}$ of A in $\partial\mathbf{R}^n_+$. The next three lemmas will be given without proofs. For proofs of 5.20, 5.21, 5.22 see [G6], [Z1], and [VU3], respectively.

5.20. Lemma. *Let E and F be disjoint compact sets in $\overline{\mathbf{R}}^n_+$ and let E^* and F^* be the symmetric images of E and F in $\partial\mathbf{R}^n_+$. If Γ_1 and Γ_2 are the families of curves joining E to F in $\mathbf{R}^n_+$ and $E \cup E^*$ to $F \cup F^*$ in $\overline{\mathbf{R}}^n$, respectively, then*

$$\mathsf{M}_p(\Gamma_2) = 2\mathsf{M}_p(\Gamma_1) .$$

5.21. Lemma. *If (Γ_j) is an increasing sequence of curve families, i.e. $\Gamma_j \subset \Gamma_{j+1}$, $j = 1,2,\ldots$, and $p > 1$, then*

$$\lim_{j\to\infty} \mathsf{M}_p(\Gamma_j) = \mathsf{M}_p\big(\bigcup \Gamma_j\big) .$$

Applying this lemma one can prove the following symmetry property of the modulus.

5.22. Lemma. *Let $p > 1$ and let E and F be subsets of $\mathbf{R}^n_+$. Then*

$$\mathsf{M}_p\big(\Delta(E,F;\mathbf{R}^n_+)\big) \ge \tfrac{1}{2}\,\mathsf{M}_p\big(\Delta(E,F)\big) .$$

5.23. Corollary. *Let E and F be sets in $\overline{\mathbf{R}}^n$ with $q(\overline{E},\overline{F}) \ge a > 0$. Then $\mathsf{M}\big(\Delta(E,F)\big) \le c(n,a) < \infty$.*

Proof. By the hypothesis there exists a ball $Q(z,r)$ in $\overline{\mathbf{R}}^n \setminus (E \cup F)$ with $r < 1/\sqrt{2}$ and with the spherical diameter $q(Q(z,r)) = a$. By easy computation (see 1.25(3)) $q(Q(z,r)) = 2r\sqrt{1-r^2}$ and hence $r \geq \frac{1}{2}a$. Let $\tilde{z}$ be the antipodal point defined in (1.16) and t a spherical isometry with $t(\tilde{z}) = 0$ as defined in (1.46). By (1.23) tE, $tF \subset Q(0, \sqrt{1-r^2})$. Because t is a spherical isometry we obtain

$$d(tE,tF) \geq q(tE,tF) = q(E,F) \geq a .$$

Next observe that (see 1.25(1))

$$Q(0, \sqrt{1-r^2}) = B^n(\sqrt{r^{-2}-1}) = B .$$

These last two relations together with 5.22, 5.17, and 5.5 yield

$$\begin{aligned} \mathsf{M}(\Delta(E,F)) &= \mathsf{M}(t(\Delta(E,F))) = \mathsf{M}(\Delta(tE,tF)) \leq 2\,\mathsf{M}(\Delta(tE,tF;B)) \\ &\leq 2a^{-n}\Omega_n(r^{-2}-1)^{n/2} \leq 2a^{-n}\Omega_n\Big(\Big(\frac{2}{a}\Big)^2 - 1\Big)^{n/2} , \end{aligned}$$

as desired. □

5.24. Lemma. *Let* $\Gamma_1, \Gamma_2, \ldots$ *be separate curve families in* $\overline{\mathbf{R}}^n$ *with* $\Gamma_j < \Gamma$ *for all* $j = 1, 2, \ldots$. *If* $p > 1$, *then*

$$\mathsf{M}_p(\Gamma)^{1/(1-p)} \geq \sum_{j=1}^{\infty} \mathsf{M}_p(\Gamma_j)^{1/(1-p)} .$$

Proof. Let $\{E_j\}$ be a family of disjoint Borel sets associated with the collection $\{\Gamma_j\}$, let $E = \bigcup E_j$, and let χ_{E_j} be the characteristic function of E_j. Fix $\rho_j \in \mathcal{F}(\Gamma_j)$ and set $\sigma_j = \rho_j \chi_{E_j}$. Then it is easy to see that $\sigma_j \in \mathcal{F}(\Gamma_j)$. Now choose a sequence (a_j) so that $a_j \in [0,1]$ and $\sum a_j = 1$ and define a Borel function ρ by

$$\rho = \sum_{j=1}^{\infty} a_j \sigma_j = \sum_{j=1}^{\infty} a_j \rho_j \chi_{E_j} .$$

We show that $\rho \in \mathcal{F}(\Gamma)$. Fix a locally rectifiable $\gamma \in \Gamma$ and for each j a subcurve $\gamma_j \in \Gamma_j$. We obtain

$$\begin{aligned} \int_\gamma \rho\, ds &= \int_\gamma \Big(\sum_j a_j \sigma_j\Big)\, ds = \sum_j a_j \int_\gamma \sigma_j\, ds \\ &\geq \sum_j a_j \int_{\gamma_j} \sigma_j\, ds \geq \sum_j a_j = 1 . \end{aligned}$$

Hence $\rho \in \mathcal{F}(\Gamma)$ and we obtain

$$\begin{aligned} \mathsf{M}_p(\Gamma) &\le \int_{\mathbf{R}^n} \rho^p \, dm = \int_E \rho^p \, dm = \sum_{j=1}^{\infty} \int_{E_j} \Big(\sum_i a_i \rho_i \chi_{E_i} \Big)^p dm = \sum_{j=1}^{\infty} \int_{E_j} a_j^p \rho_j^p \, dm \\ &\le \sum_{j=1}^{\infty} \int_{E_j} \sum_j a_j^p \rho_j^p \, dm = \int_{\mathbf{R}^n} \sum_j a_j^p \rho_j^p \, dm \le \sum_{j=1}^{\infty} a_j^p \int_{\mathbf{R}^n} \rho_j^p \, dm \, . \end{aligned}$$

Taking the infimum yields

$$\mathsf{M}_p(\Gamma) \le \sum_{j=1}^{\infty} a_j^p \, \mathsf{M}_p(\Gamma_j) \, . \tag{5.25}$$

We now apply this last inequality to prove the assertion. Clearly we may assume $\mathsf{M}_p(\Gamma) > 0$, which implies by 5.3 that $\mathsf{M}_p(\Gamma_j) \ge \mathsf{M}_p(\Gamma) > 0$. (If $\mathsf{M}_p(\Gamma) = 0$, the left side of the inequality is ∞ and there is nothing to prove.) We may also assume that $\mathsf{M}_p(\Gamma_j) < \infty$ for all j, because

$$\sum_{j=1}^{\infty} \mathsf{M}_p(\Gamma_j)^{1/(1-p)} = \sum_{j=1}^{\infty}{}^{*} \, \mathsf{M}_p(\Gamma_j)^{1/(1-p)}$$

where $\sum^*$ refers to summation over terms with $\mathsf{M}_p(\Gamma_j) < \infty$. Denote

$$t_k = \Big(\sum_{j=1}^{k} \mathsf{M}_p(\Gamma_j)^{1/(1-p)} \Big)^{-1} , \quad a_j = \mathsf{M}_p(\Gamma_j)^{1/(1-p)} t_k$$

for $j = 1, \ldots, k$ and $k = 1, 2, \ldots$ whence

$$\sum_{j=1}^{k} a_j = 1 \, .$$

The above inequality (5.25) (with $a_j = 0$ for $j \ge k+1$) gives

$$\mathsf{M}_p(\Gamma) \le t_k^p \sum_{j=1}^{k} \mathsf{M}_p(\Gamma_j)^{p/(1-p)} \mathsf{M}_p(\Gamma_j) = \Big(\sum_{j=1}^{k} \mathsf{M}_p(\Gamma_j)^{1/(1-p)} \Big)^{1-p} .$$

Letting $k \to \infty$ yields the desired result. ❐

As an example of application we consider the following simple particular case of 5.24. Let $r_1 = 1 < r_2 < \ldots < r_j < \ldots < a$ and

$$\Gamma = \Delta(S^{n-1}, S^{n-1}(a)) \, , \;\; \Gamma_i = \Delta(S^{n-1}(r_i), S^{n-1}(r_{i+1}))$$

for $i = 1, 2, \ldots$. It follows from (5.14) that in this particular case 5.24 yields (when $p = n$)

$$\log a \geq \sum_{j=1}^{\infty} \log \frac{r_{i+1}}{r_i} = \log a_0$$

where $a_0 = \lim r_j \leq a$. If we choose the sequence (r_j) so that $a_0 = a$, then equality holds. Hence 5.24 is sharp.

For the proof of the next result the reader is referred to [MO, p. 82], [G1, pp. 514–515].

5.26. Lemma. *Let* $s \in (0,1)$ *and*

$$\Gamma_1 = \Delta([0, se_1], S^{n-1}; \mathbf{B}^n) \ , \ \Gamma_2 = \Delta([0, se_1], [\tfrac{1}{s}e_1, \infty); \mathbf{R}^n) \ .$$

Then $\mathsf{M}_p(\Gamma_1) = 2^{p-1}\mathsf{M}_p(\Gamma_2)$ *for* $p > 1$.

The next result will have interesting applications later on in this book. This result was conjectured by the author and a proof was supplied by F. W. Gehring ([VU10, 2.58]).

5.27. Lemma. *Let* $\Delta_1 = \Delta([0, e_1], [t^2 e_1, \infty))$ *and* $\Delta_2 = \Delta([0, e], [t^2 e_1, \infty))$ *where* $e \in S^{n-1}$ *and* $t > 1$. *Then* $\mathsf{M}(\Delta_2) \leq \mathsf{M}(\Delta_1)$.

Proof. Denote $\Delta_{11} = \Delta([0, e_1], S^{n-1}(t))$, $\Delta_{21} = \Delta([0, e], S^{n-1}(t))$, and $\Delta_{12} = \Delta_{22} = \Delta(S^{n-1}(t), [t^2 e_1, \infty))$. Obviously

$$\mathsf{M}(\Delta_{11}) = \mathsf{M}(\Delta_{21}) \ , \ \mathsf{M}(\Delta_{12}) = \mathsf{M}(\Delta_{22}) \ .$$

Let f be the inversion in $S^{n-1}(t)$. Because $\Delta_{12} = f\Delta_{11}$ we obtain by 5.17

$$\mathsf{M}(\Delta_{11}) = \mathsf{M}(f\Delta_{11}) = \mathsf{M}(\Delta_{12}) \ .$$

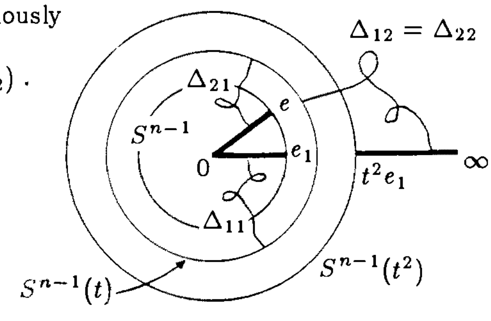


Diagram 5.3.

Next, 5.24 yields

$$\begin{aligned}\mathsf{M}(\Delta_2)^{1/(1-n)} &\geq \mathsf{M}(\Delta_{21})^{1/(1-n)} + \mathsf{M}(\Delta_{22})^{1/(1-n)} \\ &= 2\,\mathsf{M}(\Delta_{11})^{1/(1-n)}\end{aligned}$$

while the fact that Δ_1 is symmetric yields by 5.26

$$\mathsf{M}(\Delta_{11}) = 2^{n-1}\mathsf{M}(\Delta_1) \ .$$

The desired inequality follows from the last two relations. □

The family of all non-constant curves passing through a fixed point is n-exceptional as was pointed out in the paragraph following (5.15). One can show that such a family is not p-exceptional if $p > n$ (see [GOR, Chapter 3], [MAZ2]). We shall require this result in the following form, which is sometimes called the *spherical cap inequality*. For this result we introduce first an extension of the definition (5.1) of the p-modulus. Suppose that S is a euclidean sphere in $\mathbf{R}^n$ with radius r and Γ is a family of curves in S. We equip S with the restriction of the euclidean metric of $\mathbf{R}^n$ to S and with the $(n-1)$-dimensional Hausdorff measure m_{n-1} with $m_{n-1}(S) = \omega_{n-1} r^{n-1}$. Let $\mathcal{A}(\Gamma)$ be the set of all non-negative Borel-measurable functions $\rho\colon S \to \mathbf{R} \cup \{\infty\}$ with

$$\int_\gamma \rho\, ds \geq 1$$

for all locally rectifiable (with respect to the metric ds) curves γ in Γ and set

$$\mathsf{M}_n^S(\Gamma) = \inf_{\rho \in \mathcal{A}(\Gamma)} \int_S \rho^n dm_{n-1} .$$

For $\varphi \in (0, \pi)$ let $C(\varphi) = \{ z \in \mathbf{R}^n : z \cdot e_n \geq |z| \cos \varphi \}$.

5.28. Lemma. *Let* $S = S^{n-1}(r)$, $\varphi \in (0, \pi]$, *let* K *be the spherical cap* $S \cap C(\varphi)$, *and let* E *and* F *be non-empty subsets of* K.

(1) *Then*

$$\mathsf{M}_n^S(\Delta(E, F; K)) \geq \frac{b_n}{r}$$

where b_n *is a positive number depending only on* n.

(2) *If* $K = S$, *i.e.* $\varphi = \pi$, *then* b_n *may be replaced by* $c_n = 2^n b_n$ *in the above inequality.*

The proof of 5.28 (see [V7, 10.9]) is based on an application of Hölder's inequality and Fubini's theorem. A similar method yields also the following improved form of 5.28 ([R12, p. 57, Lemma 3.1], [GV1, p. 20, Lemma 3.8]).

5.29. Lemma. *Assume that* E, F, *and* K *are as in* 5.28(1). *If* $\varphi \in (0, \frac{1}{2}\pi)$, *then*

$$\mathsf{M}_n^S(\Delta(E, F; K)) \geq \frac{d_n}{\varphi\, r}$$

where d_n *depends only on* n.

5.30. Remark. Throughout the book we will denote by c_n the number in 5.28(2). The number $b_n = 2^{-n} c_n$ has the following expression

$$(5.31)\qquad \begin{cases} b_n = 2^{1-2n}\,\omega_{n-2}\, I_n^{1-n}\ , & b_2 = \dfrac{1}{2\pi}\ ,\\ I_n = \displaystyle\int_0^{\pi/2} \sin^{\frac{2-n}{n-1}} t\, dt\ . \end{cases}$$

Because $\frac{2}{\pi} t \le \sin t \le t$ for $0 \le t \le \frac{1}{2}\pi$, it follows from (5.31) that

$$(n-1)\left(\frac{\pi}{2}\right)^{1/(n-1)} \le I_n \le (n-1)\frac{\pi}{2}$$

for $n \ge 2$. One can show that $2^n c_n \to 0$ when $n \to \infty$ [AVV3].

By (5.1), any admissible function ρ yields an upper bound for $\mathsf{M}_p(\Gamma)$, that is $\mathsf{M}_p(\Gamma) \le \int_{\mathbf{R}^n} \rho^p\, dm$. The problem of finding lower bounds for $\mathsf{M}_p(\Gamma)$ is much more difficult because then we need a lower bound for $\int_{\mathbf{R}^n} \rho^p dm$ for every admissible ρ. The next important lower bound for the modulus follows by integration from 5.28 and 5.29.

5.32. Lemma. *Let $0 < a < b$ and let E, F be sets in $\mathbf{R}^n$ with*

$$E \cap S^{n-1}(t) \ne \emptyset \ne F \cap S^{n-1}(t)$$

for $t \in (a,b)$. Then

$$\mathsf{M}\big(\Delta(E,F;B^n(b)\setminus B^n(a))\big) \ge c_n \log\frac{b}{a}\ .$$

Equality holds if $E = (ae_1, be_1)$, $F = (-be_1, -ae_1)$.

5.33. Corollary. *If E and F are non-degenerate continua with $0 \in E \cap F$ then $\mathsf{M}\big(\Delta(E,F)\big) = \infty$.*

Proof. Apply 5.32 with a fixed b such that $S^{n-1}(b) \cap E \ne \emptyset \ne S^{n-1}(b) \cap F$ and let $a \to 0$. □

We next give a typical application of Lemma 5.32. Unlike 5.32 this application fails to give a sharp bound, but it yields adequate bounds in many cases (see e.g. Section 6). A sharp version of 5.34, which requires some information about spherical symmetrization, will be given in Section 7 (see 7.32 and 7.33).

5.34. Lemma. *Let $t > r > 0$ and let $E \subset B^n(r)$ be a connected set containing at least two points. Then*

$$\mathsf{M}(\Delta(S^{n-1}(t), E)) \geq c_n \log \frac{2t + d(E)}{2t - d(E)} .$$

Proof. Fix $u, v \in \overline{E}$ with $|u - v| = d(E) = d$ and choose $h \in \mathcal{GM}(B^n(t))$ with $h(u) = -se_1 = -h(v)$. By (2.27)

$$d(E) = |u - v| \leq 2 \operatorname{th} \tfrac{1}{4}\rho(u, v) = 2 \operatorname{th} \tfrac{1}{4}\rho(h(u), h(v)) = 2s ,$$

where ρ refers to the hyperbolic metric of $B^n(t)$. Applying 5.32 to the annulus $B^n(te_1, t + s) \setminus \overline{B}^n(te_1, t - s)$ with $E = hE$ and $F = S^{n-1}(t)$ we obtain

$$\begin{aligned} \mathsf{M}(\Delta(S^{n-1}(t), E)) = \mathsf{M}(\Delta(S^{n-1}(t), hE)) &\geq c_n \log \frac{t + s}{t - s} \\ &\geq c_n \log \frac{2t + d(E)}{2t - d(E)} . \quad \square \end{aligned}$$

We shall frequently apply the following lemma when proving lower bounds for the moduli of curve families. This lemma will be called the *comparison principle* for the modulus. In the applications of this lemma, the sets F_3 and F_4 will often be chosen to be non–degenerate continua (that is continua containing at least two distinct points) while the sets F_1 and F_2 will usually be very "small" sets when compared to F_3 and F_4 .

5.35. Lemma. *Let G be a domain in $\overline{\mathbf{R}}^n$, let $F_j \subset G$, $j = 1, 2, 3, 4$, and let $\Gamma_{ij} = \Delta(F_i, F_j; G)$, $1 \leq i, j \leq 4$. Then*

$$\mathsf{M}(\Gamma_{12}) \geq 3^{-n} \min\{ \mathsf{M}(\Gamma_{13}),\ \mathsf{M}(\Gamma_{24}),\ \inf \mathsf{M}(\Delta(|\gamma_{13}|, |\gamma_{24}|; G)) \} ,$$

where the infimum is taken over all rectifiable curves $\gamma_{13} \in \Gamma_{13}$ and $\gamma_{24} \in \Gamma_{24}$.

Proof. By 5.2(1) we may assume that $F_j \neq \emptyset$, $j = 1, 2, 3, 4$. Fix $\rho \in F(\Gamma_{12})$. If

$$\int_{\gamma_{13}} \rho \, ds \geq \tfrac{1}{3} \tag{5.36}$$

for every rectifiable $\gamma_{13} \in \Gamma_{13}$ or

$$\int_{\gamma_{24}} \rho \, ds \geq \tfrac{1}{3} \tag{5.37}$$

for every rectifiable $\gamma_{24} \in \Gamma_{24}$, then it follows from 5.8 and (5.1) that

$$\int_{\mathbf{R}^n} \rho^n dm \geq 3^{-n} \min\{ \mathsf{M}(\Gamma_{13}),\ \mathsf{M}(\Gamma_{24}) \} . \tag{5.38}$$

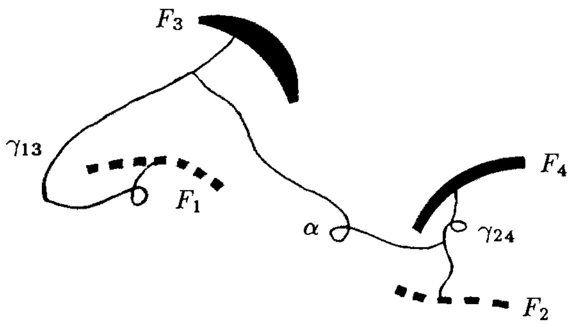


Diagram 5.4.

If both (5.36) and (5.37) fail to hold we select rectifiable curves $\gamma_{13} \in \Gamma_{13}$ and $\gamma_{24} \in \Gamma_{24}$. Because $\rho \in \mathcal{F}(\Gamma_{12})$ it follows that

$$\int_{\gamma_{13} \cup \alpha \cup \gamma_{24}} \rho\, ds \geq 1$$

for every locally rectifiable $\alpha \in \Delta = \Delta(|\gamma_{13}|, |\gamma_{24}|; G)$. Because both (5.36) and (5.37) fail to hold it follows from the last inequality that

$$\int_{\alpha} \rho\, ds \geq \tfrac{1}{3}$$

for each locally rectifiable $\alpha \in \Delta$. Hence

$$\int_{\mathbf{R}^n} \rho^n dm \geq 3^{-n} \mathsf{M}(\Delta) \geq 3^{-n} \inf \mathsf{M}\big(\Delta(|\gamma_{13}|, |\gamma_{24}|; G)\big) \tag{5.39}$$

where the infimum is taken over all rectifiable curves $\gamma_{13} \in \Gamma_{13}$ and $\gamma_{24} \in \Gamma_{24}$. In every case either (5.38) or (5.39) holds, and the desired inequality follows. ❐

5.40. Corollary. *Let* $F_j \subset \overline{\mathbf{R}}^n$ *and* $\Gamma_{ij} = \Delta(F_i, F_j)$, $1 \leq i,j \leq 4$. *Then*

$$\mathsf{M}(\Gamma_{12}) \geq 3^{-n} \min\{ \mathsf{M}(\Gamma_{13}),\ \mathsf{M}(\Gamma_{24}),\ \delta_n(r) \}$$

where $r = \min\{ q(F_1, F_3),\ q(F_2, F_4) \}$ *and*

$$\delta_n(r) = \inf \mathsf{M}\big(\Delta(E, F)\big) .$$

Here the infimum is taken over all continua E, F *in* $\overline{\mathbf{R}}^n$ *such that* $q(E) \geq r$, $q(F) \geq r$.

It is clear that $\delta_n(0) = 0$ in 5.40. In fact, this follows from 5.18(2) if we choose $r \in (0, 1/\sqrt{2})$, set $s = \sqrt{1-r^2}$, and let $r \to 0$. We are going to show that $\delta_n(r) > 0$ for $r > 0$. To this end the following corollary will be needed.

5.41. Corollary. *If* $x \in \mathbf{R}^n$, $0 < a < b < \infty$, *and* $F_1, F_2 \subset B^n(x,a)$, $F_3 \subset \mathbf{R}^n \setminus B^n(x,b)$, $\Gamma_{ij} = \Delta(F_i, F_j)$, *then*

$$\mathsf{M}(\Gamma_{12}) \geq 3^{-n} \min\Big\{ \mathsf{M}(\Gamma_{13}),\ \mathsf{M}(\Gamma_{23}),\ c_n \log\frac{b}{a} \Big\}, \tag{1}$$

$$\mathsf{M}(\Gamma_{12}) \geq d(n, b/a) \min\{ \mathsf{M}(\Gamma_{13}),\ \mathsf{M}(\Gamma_{23}) \}. \tag{2}$$

Proof. We apply the comparison principle 5.35 with $G = \mathbf{R}^n$ and $F_3 = F_4$ to get a lower bound for $\mathsf{M}(\Gamma_{12})$. It follows from 5.32 that the infimum in the lower bound of 5.35 is at least $c_n \log\frac{b}{a}$ and thus (1) follows. For the proof of (2) we observe that by 5.3 and (5.14)

$$\max\{ \mathsf{M}(\Gamma_{13}),\ \mathsf{M}(\Gamma_{23}) \} \leq A = \omega_{n-1}\Big(\log\frac{b}{a}\Big)^{1-n}.$$

By part (1) we get

$$\begin{aligned}\mathsf{M}(\Gamma_{12}) &\geq 3^{-n} \min\Big\{ \mathsf{M}(\Gamma_{13}),\ \mathsf{M}(\Gamma_{23}),\ \frac{1}{A}\Big(c_n \log\frac{b}{a}\Big) \min\{ \mathsf{M}(\Gamma_{13}),\ \mathsf{M}(\Gamma_{23}) \}\Big\} \\ &\geq d(n, b/a) \min\{ \mathsf{M}(\Gamma_{13}),\ \mathsf{M}(\Gamma_{23}) \}\end{aligned}$$

where $d(n, b/a) = 3^{-n} \min\{ 1,\ \frac{1}{A} c_n \log(b/a)\}$. □

5.42. Lemma. *For* $n \geq 2$ *there are positive numbers* d *and* D *with the following properties.*

(1) *If* $E, F \subset B^n(s)$ *are connected and* $d(E) \geq st$, $d(F) \geq st$, *then* $\mathsf{M}(\Delta(E,F)) \geq dt$.

(2) *If* $E, F \subset \overline{\mathbf{R}}^n$ *are connected and* $q(E) \geq t$, $q(F) \geq t$, *then* $\mathsf{M}(\Delta(E,F)) \geq \delta_n(t) \geq Dt$.

Proof. (1) By 5.34 we obtain

$$\mathsf{M}(\Delta(S^{n-1}(2s), E)) \geq c_n \log\frac{4s+ts}{4s-ts} \geq \tfrac{1}{2} c_n (\log 2) t$$

and similarly $\mathsf{M}(\Delta(S^{n-1}(2s), F)) \geq \frac{1}{2} c_n (\log 2) t$. Applying 5.41(1) with $F_1 = F$, $F_2 = E$, and $F_3 = S^{n-1}(2s)$ and the above estimates we get

$$\mathsf{M}(\Gamma_{12}) \geq 3^{-n} \min\{ \tfrac{1}{2} c_n (\log 2) t,\ c_n \log 2 \} \geq dt$$

where $d = \frac{1}{2} \cdot 3^{-n} c_n \log 2$.

(2) Observe first that both the first and last expressions in the asserted inequality remain invariant under spherical isometries (see 5.17). By performing a preliminary spherical isometry if necessary we may assume that $-re_1 \in E$, $re_1 \in F$, and $r \in [0,1]$ (cf. 1.25(1)). Let E_1 (F_1) be that component of $E \cap \overline{B}^n(2)$ (of $F \cap \overline{B}^n(2)$, resp.) which contains $-re_1$ (re_1). Then

$$d(E_1) \geq q(E_1) \geq \min\{\, t,\ q(S^{n-1}, S^{n-1}(2))\,\} \geq t/\sqrt{10}\,,$$

and likewise $d(F_1) \geq t/\sqrt{10}$. The proof of (2) follows from (1) with $D = d/\sqrt{10}$. □

By means of spherical symmetrization, which will be introduced in Section 7, one can give a different proof of 5.42(1) (see 7.38).

5.43. Exercise. Let E and F be non-degenerate continua in $\mathbf{B}^n$. Find a lower bound for $\mathsf{M}(\Delta(E,F;\mathbf{B}^n))$ in terms of n, $\rho(E)$, $\rho(F)$, and $\rho(E,F)$. [Hint: Fix $a_1 \in E$, $a_2 \in F$ with $\rho(a_1,a_2) = \rho(E,F)$ and let $x \in J[a_1,a_2]$ be such that $\rho(a_1,x) = \frac{1}{2}\rho(E,F)$. Let $T_x \in \mathcal{M}(\mathbf{B}^n)$ be as defined in 1.34. By conformal invariance 5.17

$$\mathsf{M}(\Delta(E,F;\mathbf{B}^n)) = \mathsf{M}(\Delta(T_xE, T_xF;\mathbf{B}^n))\,.$$

Now one can find a lower bound for the euclidean diameters $d(T_xE)$, $d(T_xF)$ in terms of $\rho(E)$, $\rho(F)$, and $\rho(E,F)$, see (2.23)–(2.25). After this apply 5.41 with $a = 1$, $b = 2$, $F_1 = T_xE$, $F_2 = T_xF$, and $F_3 = S^{n-1}(2)$. The desired result follows now from a symmetry property of the modulus, see 5.22.]

5.44. Exercise. For $E \subset \mathbf{R}^n$, $x \in \mathbf{R}^n$, and $0 < r < t$ set

$$\begin{aligned} M_t(E,r,x) &= \mathsf{M}(\Delta(S^{n-1}(x,t),\, E \cap \overline{B}^n(x,r)))\,, \\ M(E,r,x) &= M_{2r}(E,r,x)\,. \end{aligned} \tag{5.45}$$

It follows from 5.3 that $M_t(E,r,x) \leq M_s(E,r,x)$ for $0 < r < s \leq t$. Also a converse inequality is true:

$$M_t(E,r,x) \leq M_s(E,r,x) \leq a(n,r,s,t) M_t(E,r,x) \tag{5.46}$$

where a depends only on the parameters indicated. Prove (5.46) by applying 5.35 with $F_1 = E \cap \overline{B}^n(x,r)$, $F_2 = S^{n-1}(x,t)$, $F_3 = S^{n-1}(x,s)$, $F_4 = S^{n-1}(x,r)$.

5.47. Remark. The method described above fails to give the best possible constant a in (5.46). The sharp result, due to Martio and Sarvas [MS1] yields the inequality

$$M_s(E,r,x) \le b^{n-1} M_t(E,r,x)\ ,\quad b = \frac{\log(t/r)}{\log(s/r)} \tag{5.48}$$

for $0 < r < s \le t$. Here equality holds for $E = \overline{B}^n(x,r)$. The proof of (5.48) makes use of a radial quasiconformal mapping [V7, p. 49] of $B^n(x,s) \setminus \overline{B}^n(x,r)$ onto $B^n(x,t) \setminus \overline{B}^n(x,r)$.

5.49. The modulus of a ring. A domain D in $\overline{\mathbf{R}}^n$ is termed a *ring*, if $\overline{\mathbf{R}}^n \setminus D$ has exactly two components. If the components are C_0 and C_1 we write $D = R(C_0, C_1)$. The *(conformal) modulus* of a ring $R(C_0, C_1)$ is defined by

$$\operatorname{mod} R(C_0, C_1) = \Big(\frac{\mathsf{M}(\Delta(C_0, C_1))}{\omega_{n-1}}\Big)^{1/(1-n)}\ . \tag{5.50}$$

The *capacity* of $R(C_0, C_1)$ is $\mathsf{M}(\Delta(C_0, C_1))$.

A ring is a special case of a condenser, which we shall define in Section 7. In the two–dimensional case the modulus of a ring R has the following geometric interpretation: $\operatorname{mod} R = t$ if and only if R can be mapped conformally onto the annulus $\{ z \in \mathbf{R}^2 : 1 < |z| < e^t \}$. Owing to this geometric interpretation the modulus of a ring is often convenient to use in the two–dimensional case. In the multidimensional case there is no such geometric interpretation for the modulus of a ring because of the rigidity of the class of conformal mappings in $\mathbf{R}^n$, $n \ge 3$ (cf. 1.54). On the other hand there is also a geometric way of looking at the capacity of a particular ring, the so–called Grötzsch ring, which is applicable to all dimensions $n \ge 2$ (see (5.52) and (7.31)). For this reason we shall prefer the capacity to the modulus of a ring.

5.51. The Grötzsch and Teichmüller rings. The complementary components of the *Grötzsch ring* $R_{G,n}(s)$ in $\mathbf{R}^n$ are $\overline{\mathbf{B}}^n$ and $[se_1, \infty]$, $s > 1$, while those of the *Teichmüller ring* $R_{T,n}(s)$ are $[-e_1, 0]$ and $[se_1, \infty]$, $s > 0$. We shall need two special functions $\gamma_n(s)$, $s > 1$, and $\tau_n(s)$, $s > 0$, to designate the moduli of the families of all those curves which connect the complementary components of the Grötzsch and Teichmüller rings in $\mathbf{R}^n$, respectively.

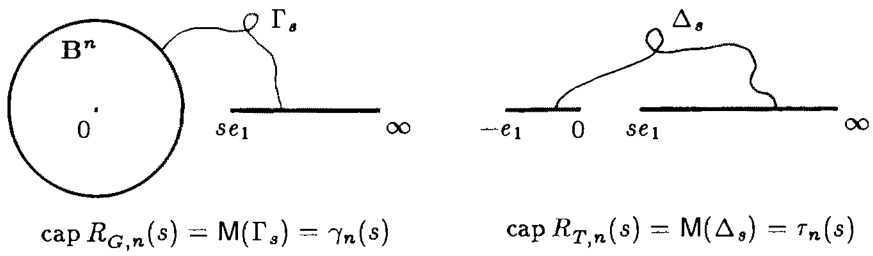


Diagram 5.5.

$$\begin{cases} \gamma_n(s) = \mathsf{M}(\Gamma_s) = \gamma(s) , \\ \tau_n(s) = \mathsf{M}(\Delta_s) = \tau(s) . \end{cases} \tag{5.52}$$

The subscript n is omitted if there is no danger of confusion. We shall refer to these functions as the Grötzsch capacity and the Teichmüller capacity.

5.53. Lemma. *For* $s > 1$, $\gamma_n(s) = 2^{n-1}\tau_n(s^2 - 1)$. *The functions* γ_n *and* τ_n *are decreasing. Furthermore,* $\lim_{s\to 1+} \gamma_n(s) = \infty$ *and* $\lim_{s\to\infty} \gamma_n(s) = 0$.

Proof. Let $\Gamma_1 = \Delta([0, \frac{1}{s}e_1], [se_1, \infty])$, $\Gamma_2 = \Delta([0, \frac{1}{s}e_1], S^{n-1})$, $\Gamma_3 = \Delta(S^{n-1}, [se_1, \infty])$. It follows from conformal invariance that $\mathsf{M}(\Gamma_2) = \mathsf{M}(\Gamma_3) = \gamma_n(s)$ and from 5.26 that

$$\gamma_n(s) = 2^{n-1}\mathsf{M}(\Gamma_1) = 2^{n-1}\tau(s^2 - 1) .$$

For each fixed $n \geq 2$ the functions γ_n and τ_n are decreasing as follows easily from 5.2(3). The limit values of γ_n follow from 5.32 and (5.14). $\square$

For the sake of completeness we set $\gamma_n(1) = \tau_n(0) = \infty$ and $\gamma_n(\infty) = \tau_n(\infty) = 0$.

5.54. Exercise. Show that

(1) $\mathsf{M}\big(\Delta([re_1, se_1], [te_1, ue_1])\big) = \tau\Big(\dfrac{(s-t)(r-u)}{(r-s)(t-u)}\Big)$, $r < s < t < u$,

(2) $\mathsf{M}\big(\Delta(S^{n-1}, [se_1, te_1])\big) = \gamma\Big(\dfrac{st-1}{t-s}\Big)$, $1 < s < t < \infty$.

5.55. Elliptic integrals and $\gamma_2(s)$. The plane Grötzsch ring can be mapped onto an annulus by an elliptic function [BF]. As shown in [HE1], [LV2, II.2]

$$\gamma_2(s) = \frac{2\pi}{\mu(1/s)} \tag{5.56}$$

for $s > 1$ where

$$\mu(r) = \frac{\pi}{2}\,\frac{K(\sqrt{1-r^2}\,)}{K(r)}\ ,\ K(r) = \int_0^1 [(1-x^2)(1-r^2x^2)]^{-1/2}dx$$

for $0 < r < 1$. The function $K(r)$ is called a complete elliptic integral of the first kind and its values can be found in tables [AS], [BF]. The modulus $\mu(r)$ satisfies the following three functional identities

$$\text{(5.57)}\qquad \begin{cases} \mu(r)\mu(\sqrt{1-r^2}) = \frac{1}{4}\pi^2\ , \\ \mu(r)\mu\Big(\dfrac{1-r}{1+r}\Big) = \frac{1}{2}\pi^2\ , \\ \mu(r) = 2\mu\Big(\dfrac{2\sqrt{r}}{1+r}\Big)\ . \end{cases}$$

From these one can derive several estimates for $\mu(r)$ [LV2, p. 62]. By [LV2, p. 62] the following inequalities hold

$$\text{(5.58)}\qquad \log\frac{1}{r} < \log\frac{1+3\sqrt{1-r^2}}{r} < \mu(r) < \log\frac{4}{r}$$

for $0 < r < 1$. From (5.58) it follows that $\lim_{r\to 0+}\mu(r) = \infty$ whence, by virtue of the functional identities (5.57), $\lim_{r\to 1-}\mu(r) = 0$. For the sake of completeness we set $\mu(0) = \infty$ and $\mu(1) = 0$. By (5.56) and (5.57) we obtain

$$\text{(5.59)}\qquad \gamma_2(s) = \frac{4}{\pi}\,\mu\Big(\frac{s-1}{s+1}\Big)\ ,\ s > 1\ .$$

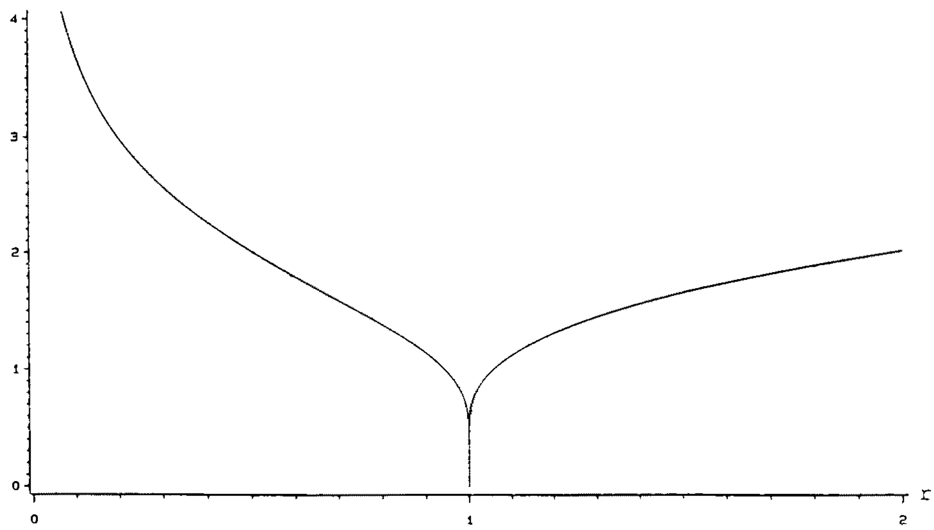


Diagram 5.6. $\mu(r)$, $0 < r \le 1$, and $\mu(1/r)$, $r > 1$ (from [AVV3]).

5.60. Exercise. Verify the following identities

(1) $$\tau_2(t) = \frac{\pi}{\mu(1/\sqrt{1+t})} = \frac{2\pi}{\mu((\sqrt{1+t}-\sqrt{t})^2)}\ ,$$

(2) $$\tau_2(t) = 2\,\tau_2\big(4\big[t+\sqrt{t(1+t)}\,\big]\big[1+t+\sqrt{t(1+t)}\,\big]\big)\ ,$$

(3) $$\mu\left(r^2\right)\,\mu\Big(\Big(\frac{1-r}{1+r}\Big)^2\Big) = \pi^2\ .$$

5.61. Exercise. In the study of distortion theory of quasiconformal mappings in Section 11 below the following special function will be useful

$$\varphi_{K,n}(r) = \frac{1}{\gamma_n^{-1}(K\gamma_n(1/r))}$$

for $0 < r < 1$, $K > 0$. (Note: Lemma 7.20 below shows that γ_n is strictly decreasing and hence that γ_n^{-1} exists.) Show that $\varphi_{AB,n}(r) = \varphi_{A,n}(\varphi_{B,n}(r))$ and $\varphi_{A,n}^{-1}(r) = \varphi_{1/A,n}(r)$ and that

$$\varphi_{K,2}(r) = \varphi_K(r) = \mu^{-1}\big(\tfrac{1}{K}\mu(r)\big)\ .$$

Verify also that

(1) $$\varphi_2(r) = \frac{2\sqrt{r}}{1+r}\ ,$$

(2) $$\varphi_K(r)^2 + \varphi_{1/K}\big(\sqrt{1-r^2}\,\big)^2 = 1\ .$$

Exploiting (1) and (2) find $\varphi_{1/2}(r)$. Show also that

(3) $$\varphi_{1/K}\Big(\frac{1-r}{1+r}\Big) = \frac{1-\varphi_K(r)}{1+\varphi_K(r)}\ ,$$

(4) $$\varphi_K\Big(\frac{2\sqrt{r}}{1+r}\Big) = \frac{2\sqrt{\varphi_K(r)}}{1+\varphi_K(r)}\ .$$

5.62. Exercise. Verify the following identities for $K, t > 0$

(1) $$\tau_2^{-1}\big(\tau_2(t)/K\big) = \frac{1}{\tau_2^{-1}(K\tau_2(1/t))}\ ,$$

(2) $$\tau_2(t) = \frac{4}{\tau_2(1/t)}\ .$$

The above functional identities, e.g. (5.57) and 5.60(2), are restricted to the two–dimensional case. For the multidimensional case $n \geq 3$ there is no explicit expression like (5.56) for $\gamma_n(s)$ or $\tau_n(s)$ and no functional identities are known for $\gamma_n(s)$ or $\tau_n(s)$ except the basic relationship 5.53. The well–known upper and lower estimates for $\gamma_n(s)$ and $\tau_n(s)$ will be given in Section 7. Next we shall show that for all dimensions $n \geq 2$ the Teichmüller capacity $\tau_n(s)$ satisfies certain *functional inequalities.*

5.63. Lemma. *The following functional inequalities hold:*

(1) $\tau(s) \le \gamma(1+2s) = 2^{n-1}\tau(4s^2+4s)$, $s>0$,

(2) $\tau(s) \le 2\tau\big(2s+2s\sqrt{1+1/s}\big)$, $s>0$,

(3) $\tau(s) \le \tau(t) + \tau\Big(\dfrac{s(1+t)}{t-s}\Big)$, $0<s<t<\infty$,

(4) $\tau(u) \le \tau\Big(\dfrac{uv}{u+v+1}\Big) \le \tau(u)+\tau(v)$, $u,v>0$.

Proof. (1) Let $\Gamma = \Delta(\,S^{n-1}(-\frac{1}{2}e_1,\frac{1}{2}),\,[se_1,\infty]\,)$. Then by 5.53

$$\mathsf{M}(\Gamma) = \gamma(1+2s) = 2^{n-1}\tau(4s^2+4s)$$

while by 5.3 $\tau(s) \le \mathsf{M}(\Gamma)$ and the desired inequality follows.

(2) We can map the Teichmüller ring $R_{T,n}(s)$ by a Möbius transformation onto a ring in $\overline{\mathbf{R}}^n$ with complementary components $[-e_1,e_1]$ and $[be_1,\infty]\cup[-be_1,\infty]$ where $b = 1+2s(1+\sqrt{1+1/s}\,)$. By a symmetry property of the modulus, Lemma 5.20, we obtain

$$\tau(s) = 2\mathsf{M}\big(\Delta(\,[0,e_1],\,[be_1,\infty];\,\{x\in\mathbf{R}^n : x_1>0\})\big) \le 2\tau(b-1)$$

as desired.

(3) Let $\Gamma_1 = \Delta(\,[-e_1,0]\,,[se_1,te_1]\,)$, $\Gamma_2 = \Delta(\,[-e_1,0]\,,[te_1,\infty]\,)$. Then by 5.54(1)

$$\tau(s) \le \mathsf{M}(\Gamma_1\cup\Gamma_2) \le \mathsf{M}(\Gamma_1)+\mathsf{M}(\Gamma_2) = \tau\Big(\frac{s(1+t)}{t-s}\Big) + \tau(t)\ .$$

(4) After a change of variables the second inequality in (4) follows from (3). The first inequality follows because τ_n is decreasing (5.53). ❒

5.64. Corollary. $\tau(s) \le 2\tau(\sqrt{s}\,) \le 2^n\tau(s)$, $s>0$.

Proof. The first and second inequality follow from 5.63(2) and 5.63(1), respectively. ❒

5.65. Remark. Corollary 5.64 applied to τ_2 yields by 5.60(1) the following two-sided inequality for the function μ :

$$\mu\big(1/\sqrt{1+t}\,\big) \le 2\mu\big(1/\sqrt{1+\sqrt{t}}\,\big) \le 4\mu\big(1/\sqrt{1+t}\,\big)\ ,\quad t>0\ ,$$

which can also be derived from the identities (5.57). The second inequality in (5.58) can be derived from the lower bound in [LV2, p. 62]

$$\log\frac{(1+\sqrt[4]{1-r^2}\,)^2}{r} < \mu(r)\ .$$

5.66. Remark. It follows from (5.58) that $\log(1/r) < \mu(r) < \log(4/r)$ for all $r \in (0,1)$ where both bounds have the correct asymptotic behavior as $r \to 0+$. For $r \to 1-$ the second inequality is very weak since $\mu(r) \to 0$ as $r \to 1-$. An improved two-sided inequality for $\mu(r)$ can be obtained as follows. Exploiting (5.58) together with the functional identity (5.57) we obtain

$$\frac{\pi^2}{4\log\frac{4}{\sqrt{1-r^2}}} < \mu(r) = \frac{\pi^2}{4\mu(\sqrt{1-r^2})} < \frac{\pi^2}{4\log\frac{1}{\sqrt{1-r^2}}} .$$

This inequality together with (5.58) implies for $r \in (0,1)$

$$(5.67) \qquad \max\left\{\log\frac{1}{r}, \frac{\pi^2}{4\log\frac{4}{\sqrt{1-r^2}}}\right\} < \mu(r) < \min\left\{\log\frac{4}{r}, \frac{\pi^2}{4\log\frac{1}{\sqrt{1-r^2}}}\right\} .$$

The asymptotic behavior in (5.67) is correct at both ends $r = 0$ and $r = 1$.

5.68. Exercise. Let A, B, C, D be distinct points on the unit circle S^1 in the stated order and 2α and 2β the lengths of the arcs AB and CD, respectively. Find the least value of $\mathsf{M}(\Delta(AB, CD))$. [Hint: $|A-C||B-D| = |A-B||C-D| + |B-C||A-D|$ by Ptolemy's theorem [CG, p. 42], [BER, 10.9.2].]

5.69. Remark. The function μ has several interesting properties which are given in [AVV3]. For instance the inequalities

$$\mu\Big(\frac{ab}{1+a'b'}\Big) < \mu(a) + \mu(b) \le \mu\Big(\frac{ab}{(1+a')(1+b')}\Big) \le 2\mu(\sqrt{ab})$$

hold for $a, b \in (0,1)$ where $a' = \sqrt{1-a^2}$. It follows from (5.57) that the second and third inequalities hold as equalities for $a = b$.

5.70. Exercise. Show that for $s > 0$ and $r \in (1, 1+s)$ the inequality

$$\tau_n(s)^{1/(1-n)} \ge \gamma_n(r)^{1/(1-n)} + \gamma_n\big((1+s)/r\big)^{1/(1-n)}$$

holds with equality if $r = \sqrt{1+s}$.

5.71. Notes. Most results in this section are standard and are well represented in the literature, e.g. in [V7]. The origin of some less standard results is indicated above in connection with each result. Next we shall make some additional remarks on the results of this section. Lemmas 5.7 and 5.24 are from [F], 5.20, 5.21, and

5.22 from [G6], [Z1], [VU3]. The comparison principle 5.35 has its roots in [MRV2, 3.11], but under this name it was introduced and developed by R. Näkki [N2] and the author [VU8]. An account of the properties of the function $\mu(r)$ is given in [LV2]. The inequalities in 5.63 and 5.64 are from [VU13]. For further properties of $\tau_n(s)$ the reader is referred to Section 7 and to [AVV3]. A highly interesting study of the complete elliptic integral $K(r)$ and Gauss' arithmetic–geometric mean is contained in [BB].

5.72. Notes. The extremal length method of L. V. Ahlfors and A. Beurling [AB] (1950) has its roots in the length–area method whose use is widespread throughout geometric function theory. Some historical comments about the origin of the length–area method are made by L. V. Ahlfors [A3, p. 50, 81] and by J. Jenkins in [JE1], [JE2]. According to Jenkins the first result of this type is due to H. Bohr in 1918 and slightly later results are due to W. Gross, G. Faber and R. Courant. In 1928 H. Grötzsch [GRÖ] published his well-known work on quasiconformal mappings and in a subsequent series of papers developed his strip method, giving applications to a variety of problems. In his dissertation in 1955 J. Hersch [HE1] established connections among harmonic measure, extremal length, and other conformal invariants. A survey of some function–theoretic applications of the extremal length is given by B. Rodin in [RO], which contains also a good bibliography of the subject. See also the bibliography in the book of G. V. Kuz'mina [KU]. In his book [O] M. Ohtsuka gives several function–theoretic applications of the extremal length.

B. Fuglede [F] was the first to consider, in 1957, the p–modulus in the multidimensional case. He also considered the modulus of a surface family as well as the modulus of a system of measures (see also P. Mattila [MAT1]). Later these notions were developed mainly in connection with the theory of quasiconformal mappings (see O. Lehto and K. I. Virtanen [LV2], F. W. Gehring [G1], [G2], [G9] and J. Väisälä [V1], [V3], [V7], [V10]). The notion of p–capacity, which is closely connected with that of p–modulus (see Section 7) has been studied by many authors in the setup of non–linear potential theory (see the references given at the end of Sections 6 and 7). The paper of C. Loewner [LO] is one of the first papers dealing with conformal capacity in space. See also the books of P. Caraman [C1, pp. 46–70] and A. V. Sychev [SY, pp. 26–35]. The book of Caraman contains a useful and very extensive bibliography.

6. The modulus as a set function

In this section we shall consider the problem of finding estimates for $\mathsf{M}(\Delta(E,F))$ when E and F are disjoint non–empty sets in $\overline{\mathbf{R}}^n$. In view of the conformal invariance of the n–modulus 5.17, one would like to find estimates which reflect this invariance property in the following way: The estimate should give the same lower/upper bound for $\mathsf{M}(\Delta(E,F))$ and $\mathsf{M}(\Delta(hE,hF))$ whenever $h \in \mathcal{GM}(\overline{\mathbf{R}}^n)$. In most estimates (see e.g. 5.42(1) or 6.1 below) this requirement is not completely met, the estimate remaining invariant only under the action of a subgroup of $\mathcal{GM}(\overline{\mathbf{R}}^n)$, e.g. under translations, stretchings, or spherical isometries. Some aspects of this problem will be discussed in Sections 7 and 8.

In the present section we shall prove the existence of a set function $c(\cdot)$, defined in the class of all subsets of $\overline{\mathbf{R}}^n$, with the following properties.

6.1. Theorem. *For $n \geq 2$ there exist positive numbers $d_1, \ldots, d_4$ and a set function $c(\cdot)$ in $\overline{\mathbf{R}}^n$ such that*

(1) $c(E) = c(hE)$ *whenever* $h\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ *is a spherical isometry and* $E \subset \overline{\mathbf{R}}^n$.

(2) $c(\emptyset) = 0$, $A \subset B \subset \overline{\mathbf{R}}^n$ *implies* $c(A) \leq c(B)$ *and* $c\big(\bigcup_{j=1}^{\infty} E_j\big) \leq d_1 \sum_{j=1}^{\infty} c(E_j)$ *if* $E_j \subset \overline{\mathbf{R}}^n$.

(3) *If* $E \subset \overline{\mathbf{R}}^n$ *is compact, then* $c(E) > 0$ *if and only if* $\operatorname{cap} E > 0$. *Moreover* $c(\overline{\mathbf{R}}^n) \leq d_2 < \infty$.

(4) $c(E) \geq d_3\, q(E)$ *if* $E \subset \overline{\mathbf{R}}^n$ *is connected and* $E \neq \emptyset$.

(5) $\mathsf{M}(\Delta(E,F)) \geq d_4 \min\{\, c(E),\, c(F)\,\}$, *if* $E, F \subset \overline{\mathbf{R}}^n$.

Furthermore, for $n \geq 2$ and $t \in (0,1)$ there exists a positive number d_5 such that

(6) $\mathsf{M}(\Delta(E,F)) \leq d_5 \min\{c(E), c(F)\}$ *whenever* $E, F \subset \overline{\mathbf{R}}^n$ *and* $q(E,F) \geq t$.

It should be emphasized that the main interest in Theorem 6.1 lies in the inequalities (5) and (6). The condition $\operatorname{cap} E > 0$ in 6.1(3) is not needed in this section and its definition will be postponed until Section 7.

We shall next give the reader some idea about the set function $c(\cdot)$. To this end define (see (5.45))

$$M_t(E,r,x) = \mathsf{M}(\Delta(S^{n-1}(x,t),\ \overline{B}^n(x,r) \cap E; \overline{\mathbf{R}}^n)) \ ,\qquad (6.2)$$
$$M(E,r,x) = M_{2r}(E,r,x)$$

whenever $E \subset \overline{\mathbf{R}}^n$, $x \in \mathbf{R}^n$, and $0 < r < t$. Moreover, let $E^{-1} = \{ x/|x|^2 : x \in E \}$ and

$$a(E) = \max\{ M(E,1,0),\ M(E^{-1},1,0) \} \qquad (6.3)$$

for $E \subset \overline{\mathbf{R}}^n$. It follows from the results of this section that there are numbers γ_1 and γ_2 depending only on the dimension n such that

$$\gamma_1 a(E) \le c(E) \le \gamma_2 a(E) \ . \qquad (6.4)$$

In what follows we shall give a construction of the set function $c(E)$. We remark that there may also be many other methods of constructing $c(E)$: it is clear by (6.4) that any method which yields a set function differing from $a(E)$ by at most a multiplicative constant is adequate also for constructing $c(E)$.

For the next lemma we recall that the balls $Q(x,r)$ of the metric space $(\overline{\mathbf{R}}^n, q)$ were defined in (1.22).

6.5. Lemma. *Let $r > 1$ and $t \in (0,1)$ be such that $q(S^{n-1}, S^{n-1}(r)) \ge 2t$. There is a number $b(t)$ depending only on n , r , and t such that the following holds. If $E \subset \overline{\mathbf{B}}^n$ and $G_t = \bigcup_{x\in E} Q(x,t)$, then*

$$\mathsf{M}(\Delta(E,\partial G_t)) \le b(t)\, \mathsf{M}(\Delta(E, S^{n-1}(r))) \ .$$

Proof. Let $F_1 = E$, $F_2 = S^{n-1}(r)$, and $F_3 = \partial G_t = F_4$. Because $q(F_1,F_3) \ge t$ and $q(F_2,F_4) \ge t$ it follows from the comparison principle 5.40 and 5.42(2) that

$$\mathsf{M}(\Gamma_{12}) \ge 3^{-n} \min\{ \mathsf{M}(\Gamma_{13}),\ \mathsf{M}(\Gamma_{23}),\ Dt \} \qquad (6.6)$$

where $\Gamma_{ij} = \Delta(F_i, F_j)$. We shall first find a lower bound for $\mathsf{M}(\Gamma_{23})$. From the choice of t it follows that $t < 1/\sqrt{2}$ and hence $q(Q(z,t)) = 2t\sqrt{1-t^2} \ge t\sqrt{2}$ (see

1.25). It follows that F_3 contains a continuum of euclidean diameter at least $t\sqrt{2}$. Hence we deduce by 5.9 and 5.34 that

$$\mathsf{M}(\Gamma_{23}) \geq c_n \log \frac{2r + t\sqrt{2}}{2r - t\sqrt{2}} \geq \frac{c_n t \log 2}{r} \geq \frac{D\,t}{r} . \tag{6.7}$$

Here D is the number in 5.42(2) and (6.6). Since $d(|\gamma|) \geq q(|\gamma|) \geq t$ and $|\gamma| \subset B^n(r)$ for $\gamma \in \Delta(F_1, F_3; G_t) = \Delta$ and $\mathsf{M}(\Delta) = \mathsf{M}(\Gamma_{13})$ (cf. (5.10)) we get by 5.5

$$\mathsf{M}(\Gamma_{13}) \leq \Omega_n \frac{r^n}{t^n} .$$

By (6.6) and (6.7)

$$\begin{aligned} \mathsf{M}(\Gamma_{12}) &\geq 3^{-n} \min\Big\{ \mathsf{M}(\Gamma_{13}) , \frac{D\,t}{r} \Big\} \\ &\geq 3^{-n} \min\Big\{ \mathsf{M}(\Gamma_{13}) , \frac{D t^{n+1}}{\Omega_n r^{n+1}} \mathsf{M}(\Gamma_{13}) \Big\} = \frac{1}{b(t)} \mathsf{M}(\Gamma_{13}) , \\ b(t) &= 3^n / \min\{ 1 , D t^{n+1}/(\Omega_n r^{n+1}) \} . \quad \square \end{aligned}$$

6.8. The construction of $c(E)$. For $E \subset \overline{\mathbf{R}}^n$, $0 < r < t < 1$ denote (cf. (6.2))

$$\begin{cases} m_t(E, r, x) = \mathsf{M}\big(\Delta(\partial Q(x,t),\ E \cap \overline{Q}(x,r))\big) , \\ m(E, x) = m_s(E, 1/\sqrt{2}, x) ; \ s = \tfrac{1}{2}\sqrt{3} . \end{cases} \tag{6.9}$$

We define (see (1.16) and (1.23))

$$\begin{cases} c(E, x) = \max\{ m(E, x) , m(E, \widetilde{x}) \} , \\ c(E) = \inf\{ c(E, x) \ : \ x \in \overline{\mathbf{R}}^n \} . \end{cases} \tag{6.10}$$

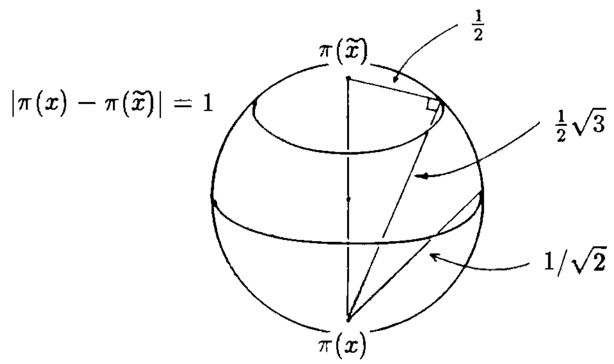


Diagram 6.1.

6.11. Remark. By 5.18(2)

$$(6.12)\qquad \begin{cases} m_t(E,r,x) \le \omega_{n-1}\left[\log\left(\dfrac{t}{r}\sqrt{\dfrac{1-r^2}{1-t^2}}\right)\right]^{1-n}, \\ m(E,x) \le m(\overline{\mathbf{R}}^n, x) = \omega_{n-1}(\log\sqrt{3})^{1-n}. \end{cases}$$

If $F \subset \overline{Q}(x,r)$, where $r \in (0, 1/\sqrt{2}]$, by (6.12) we obtain

$$(6.13)\qquad m(F,x) \le \omega_{n-1}\left[\log\left(\frac{1}{r}\sqrt{3(1-r^2)}\right)\right]^{1-n}.$$

Hence $c(F,x) \to 0$ as $r \to 0$. Note that equality holds in (6.13) if $F = \overline{Q}(x,r)$. Exploiting 1.18(1) one can simplify the upper bound in (6.12).

6.14. Lemma. *There exists a positive number* d_1 *depending only on* n *such that*

$$c(E,x) \le d_1 c(E,y)$$

for $x, y \in \overline{\mathbf{R}}^n$ *and* $E \subset \overline{\mathbf{R}}^n$. *In particular,*

$$c(E) \le c(E,x) \le d_1 c(E).$$

Proof. Let $U = Q(x, 1/\sqrt{2})$. By 5.18(2) or by (6.12) we obtain

$$(6.15)\qquad \mathsf{M}(\Delta(\partial Q(x, \tfrac{1}{2}\sqrt{3}), \partial U)) = \mathsf{M}(\Delta(\partial Q(x, \tfrac{1}{2}), \partial U)) = \omega_{n-1}(\log\sqrt{3})^{1-n} = a.$$

Fix $x, y \in \overline{\mathbf{R}}^n$. In what follows we shall assume that

$$(6.16)\qquad c(E,x) = m(E,x).$$

The other case $c(E,x) = m(E,\widetilde{x})$ can be dealt with exactly in the same way; even the constants will be the same in the other case. Let

$$E_1^* = E \cap \overline{Q}(x, 1/\sqrt{2}) \cap \overline{Q}(y, 1/\sqrt{2}),$$
$$E_2^* = (E \setminus E_1^*) \cap \overline{Q}(x, 1/\sqrt{2}).$$

It follows from 5.9 and (6.16) that either

$$2\,\mathsf{M}(\Delta(\partial V, E_1^*)) \ge c(E,x) \quad \text{or} \quad 2\,\mathsf{M}(\Delta(\partial V, E_2^*)) \ge c(E,x)$$

where $V = Q(x, \frac{1}{2}\sqrt{3})$. In the first case denote $F_1 = E_1^*$, $F_2 = \partial Q(y, \frac{1}{2}\sqrt{3})$, $F_3 = \partial V$ and $F_4 = \partial Q(y, 1/\sqrt{2})$. In the second case let $F_1 = E_2^*$, $F_2 = \partial Q(\widetilde{y}, \frac{1}{2}\sqrt{3})$, $F_3 = \partial V$, and $F_4 = \partial Q(\widetilde{y}, 1/\sqrt{2})$. In both cases (see 1.25(1) and (1.15))

$$\min\{\, q(F_1, F_3),\, q(F_2, F_4)\,\} \ge q(\partial V,\, \partial Q(x, 1/\sqrt{2}))$$
$$= q(S^{n-1}(\sqrt{3}),\, S^{n-1}) = \frac{\sqrt{3}-1}{\sqrt{8}} = \delta.$$

We obtain by 5.40, 5.42(2), and (6.15)

$$\begin{aligned} c(E,y) &\geq \mathsf{M}(\Gamma_{12}) \geq 3^{-n} \min\{\, \mathsf{M}(\Gamma_{13}),\, \mathsf{M}(\Gamma_{24}),\, D\delta \,\} \\ &\geq 3^{-n} \min\{\, \tfrac{1}{2} c(E,x)\,,\, a\,,\, D\delta \,\}\,. \end{aligned}$$

Because $c(E,x) \leq a$ by (6.12) and (6.15) we obtain from this inequality

$$c(E,y) \geq 3^{-n} \min\{\, \tfrac{1}{2} c(E,x)\,,\, D\delta \,\} \geq d_1^{-1} c(E,x)\,;$$

$$d_1^{-1} = 3^{-n} \min\{\, \tfrac{1}{2}\,,\, D\delta (\log\sqrt{3}\,)^{n-1}/\omega_{n-1} \}\,,$$

which yields the desired bound. ❐

6.17. Lemma. *If* $E \subset \overline{\mathbf{R}}^n$, *then*

$$c(E) \leq \omega_{n-1} \Big(\log \frac{\sqrt{3}}{\sqrt{2}\, q(E)} \Big)^{1-n}\,.$$

Proof. Assume first that $q(E) \geq 1/\sqrt{2}$. In this case

$$c(E) \leq c(E,0) \leq \omega_{n-1} (\log\sqrt{3}\,)^{1-n} \leq \omega_{n-1} \Big(\log \frac{\sqrt{3}}{\sqrt{2}\, q(E)} \Big)^{1-n}$$

by (6.12). Assume next that $q(E) \leq 1/\sqrt{2}$. In this case $E \subset \overline{Q}(z, q(E))$, $z \in E$, and the proof follows from (6.13). ❐

6.18. Corollary. *If* $E \subset \overline{\mathbf{R}}^n$ *is connected, then*

$$c(E) \geq d_3\, q(E)\,.$$

Proof. It follows from the definition (6.10) that $c(E) = c(hE)$ whenever h is a spherical isometry. Hence both sides of the asserted inequality remain invariant under spherical isometries. By performing an auxiliary spherical isometry if necessary we may assume that $0 \in E$. Then $E \cap \overline{\mathbf{B}}^n$ has a connected component E_1 with $0 \in E_1$ and hence by (1.15)

$$d(E_1) \geq q(E_1) \geq \min\{\, 1/\sqrt{2}\,,\, q(E)\} \geq q(E)/\sqrt{2}\,.$$

By 5.34 we obtain (see (6.2), (6.10), and 1.25(1))

$$c(E,0) \geq M_{\sqrt{3}}(E_1, 1, 0) \geq c_n \log \frac{2\sqrt{3} + q(E)/\sqrt{2}}{2\sqrt{3} - q(E)/\sqrt{2}} \geq c_n q(E)/\sqrt{6}\,.$$

The proof with $d_3 = c_n/(d_1\sqrt{6}\,)$ follows now from 6.14. ❐

6.19. Lemma. $\mathsf{M}(\Delta(E,F)) \geq d_4 \min\{c(E), c(F)\}$.

Proof. Fix $x \in \overline{\mathbf{R}}^n$. Let $z \in \{x, \tilde{x}\}$ with $m(E,z) = c(E,x)$ and denote

$$F_1 = E \cap \overline{Q}(z, 1/\sqrt{2})\ ,\ F_3 = \partial Q(z, \tfrac{1}{2}\sqrt{3})\ .$$

Let $w \in \{x, \tilde{x}\}$ be such that $m(F,w) = c(F,x)$ and denote

$$F_2 = F \cap \overline{Q}(w, 1/\sqrt{2})\ ,\ F_4 = \partial Q(w, \tfrac{1}{2}\sqrt{3})\ .$$

We see that (cf. 1.25)

$$\min\{\, q(F_1,F_3),\, q(F_2,F_4)\,\} \geq q(S^{n-1}(\sqrt{3}),\, S^{n-1}) = \frac{\sqrt{3}-1}{\sqrt{8}} = \delta\ .$$

Set $\Gamma_{ij} = \Delta(F_i, F_j)$. It follows from the comparison principle 5.40 and 5.42(2) (see also 5.9) that

$$\begin{aligned} \mathsf{M}(\Delta(E,F)) &\geq \mathsf{M}(\Gamma_{12}) \geq 3^{-n} \min\{\, c(E,x),\, c(F,x),\, D\delta\,\} \\ &\geq d_4 \min\{\, c(E,x),\, c(F,x)\,\} \geq d_4 \min\{\, c(E),\, c(F)\,\} \end{aligned}$$

where $d_4 = 3^{-n} \min\{\, 1,\, D\delta(\log\sqrt{3}\,)^{n-1}/\omega_{n-1}\}$ and the second last inequality follows from the fact that $c(E,x),\, c(F,x) \leq \omega_{n-1}(\log\sqrt{3}\,)^{1-n}$ (cf. (6.12)). □

6.20. Lemma. *Let* $E, F \subset \overline{\mathbf{R}}^n$ *be sets with* $q(E,F) \geq t > 0$. *Then*

$$\mathsf{M}(\Delta(E,F)) \leq d_5 \min\{\, c(E),\, c(F)\,\}$$

where d_5 *depends only on* n *and* t.

Proof. Let $E_1 = E \cap \overline{Q}(0, 1/\sqrt{2})$, $E_2 = E \setminus E_1$, $F_1 = F \cap \overline{Q}(0, 1/\sqrt{2})$, $F_2 = F \setminus F_1$. Let $\Gamma_1 = \Delta(E_1, F_1)$, $\Gamma_2 = \Delta(E_1, F_2)$, $\Gamma_3 = \Delta(E_2, F_1)$, and $\Gamma_4 = \Delta(E_2, F_2)$. By 5.9

$$\mathsf{M}(\Delta(E,F)) \leq 4 \max\{\, \mathsf{M}(\Gamma_j) : j = 1,2,3,4\,\}\ .$$

Without loss of generality we may assume that the maximum on the right side of this inequality is equal to $\mathsf{M}(\Gamma_2)$ because in the other cases the proof will be similar. Let

$$E_1^t = \bigcup\{\, Q(x, \tfrac{1}{8}t) : x \in E_1\,\}\ ,\ F_2^t = \bigcup\{\, Q(x, \tfrac{1}{8}t) : x \in F_2\,\}\ .$$

If $\gamma \in \Gamma_2$, then clearly $|\gamma| \cap \partial E_1^t \neq \emptyset \neq |\gamma| \cap \partial F_2^t$ and hence by 5.3

$$(6.21) \qquad \tfrac{1}{4}\,\mathsf{M}(\Delta(E,F)) \le \mathsf{M}(\Gamma_2) \le \min\{\, \mathsf{M}(\Delta(E_1,\partial E_1^t))\,,\, \mathsf{M}(\Delta(F_2,\partial F_2^t))\,\}\,.$$

We shall now find an upper bound for $\mathsf{M}(\Delta(E_1,\partial E_1^t))$. A simple calculation shows that

$$q(S^{n-1}(\sqrt{3}),\, S^{n-1}) = \frac{\sqrt{3}-1}{\sqrt{8}} > \frac{1}{4}\,.$$

Since $E_1 \subset \overline{\mathbf{B}}^n$ we get by 6.5

$$\mathsf{M}(\Delta(E_1,\partial E_1^t)) \le \mathsf{M}(\Delta(E_1,S^{n-1}(\sqrt{3})))\; b(t/8)\,,$$
$$b(t/8) = 3^n / \min\{\,1\,,\, Dt^{n+1}/((8\sqrt{3})^{n+1}\Omega_n)\}\,.$$

A similar estimate holds for $\mathsf{M}(\Delta(F_2,\partial F_2^t))$ as well. As a result we obtain in view of (6.21), (6.10), and 6.14

$$\begin{aligned}\mathsf{M}(\Delta(E,F)) &\le 4\, b(t/8) \min\{\, c(E,0)\,,\, c(F,0)\,\} \\ &\le 4\, d_1\, b(t/8) \min\{\, c(E)\,,\, c(F)\,\}\,. \quad \square\end{aligned}$$

6.22. Corollary. *If* $E,F \subset \overline{\mathbf{R}}^n$ *with* $q(E,F) \ge t > 0$, *then* $\mathsf{M}(\Delta(E,F)) \le d_6$.

Proof. By(6.12) and 6.14 $c(E) \le c(\overline{\mathbf{R}}^n) = \omega_{n-1}(\log\sqrt{3}\,)^{1-n} = d_2$. The proof with $d_6 = d_2 d_5$ follows from 6.20. $\square$

Recall that a different proof of 6.22 was given in 5.23.

Proof of Theorem 6.1. Part (1) is clear by the definition of $c(\cdot)$. Part (2) follows from (6.10), 6.14, and 5.9:

$$c\Big(\bigcup_{j=1}^{\infty} E_j\Big) \le c\Big(\bigcup_{j=1}^{\infty} E_j\,,\,0\Big) \le \sum_{j=1}^{\infty} c(E_j,0) \le d_1 \sum_{j=1}^{\infty} c(E_j)\,.$$

The other assertions in (2) follow from 5.9. The proofs of (4), (5), and (6) were given in 6.18, 6.19, and 6.20, respectively. The proof of (3) follows from (5), (6), and the definition of a set with positive capacity, which will be given in Section 7 (see 7.12). $\square$

6.23. Exercise. Find a lower bound for $c(\overline{B}^n(x,r))$.

6.24. Exercise. Applying (5.46) and the results of this section show that (6.4) holds.

6.25. Exercise. Let $E = \{0\} \cup \left(\bigcup_{k=1}^{\infty} S^{n-1}(2^{-k})\right)$ and $E(t) = \{ z \in \mathbf{R}^n : q(z,E) < t \}$. Show that $\mathsf{M}\big(\Delta(E, \partial E(t))\big) \geq \alpha t^{1-n} \log \frac{1}{t}$ for small t where α depends only on n. [Hint: Apply (5.14).] *Conclusion:* The function $b(t)$ in 6.5 must grow so fast that $b(t)t^{n-1}/\log\frac{1}{t} \not\to 0$ as $t \to 0$. From the proof of 6.5 it follows that the rate of growth of $b(t)$ is at most t^{-1-n}, and the best rate of growth will be given in 6.27.

An appendix to Section 6. In this appendix we shall carry out some computations which we shall not need later on in this book but which may be of independent interest. We are now going to prove an improved form of Lemma 6.5 and shall show that the function $b(t)$ in 6.5 can be chosen so that its rate of growth is at most t^{-n}. It follows from Exercise 6.25 that the power $-n$ cannot be replaced by $1-n$ (see also 6.28).

The following discussion is based on a Poincaré inequality type result of Yu. G. Reshetnyak [R12, p. 60, Lemma 3.3], and the proof of Lemma 6.27 below is also due to him. The author wishes to thank Yu. G. Reshetnyak for contributing this result. For the proof we need also some results from the early parts of Section 7. In particular, Lemma 7.8 will be useful.

6.26. Lemma ([R12, p. 60, Lemma 3.3]). *Let u be a function of class $C_0^\infty(\mathbf{R}^n)$ such that $u(x) = 0$ for $|x| \geq r > 0$. Then the inequality*

$$\int_{\mathbf{R}^n} |u|^n dm \leq (2r)^n \int_{\mathbf{R}^n} |\nabla u|^n dm$$

holds.

6.27. Lemma. *Let E be a compact set in $B^n(R)$ and let $E(t) = E + B^n(t)$ for $t > 0$. Then*

$$\mathsf{M}\big(\Delta(\partial E(t), E)\big) \leq a(t)\, \mathsf{M}\big(\Delta(\partial E(1), E)\big)$$

for $t > 0$ where $a(t) = a(1)$ for $t \geq 1$ and $a(t) \leq a_1 t^{-n}$ for $t \in (0,1)$, and a_1 depends only on n and R.

Proof. Fix $\epsilon > 0$. In view of (7.3)-7.8 there exists a function $u \in C_0^\infty(E(1))$ with $u(x) \geq 1$ for $x \in E$ and

$$\int_{\mathbf{R}^n} |\nabla u|^n dm \leq \epsilon + \mathsf{M}\big(\Delta(\partial E(1), E)\big) = \epsilon + \operatorname{cap}(E(1), E) \, .$$

There exists a constant b_1 depending only on n and for each $t \in (0,1]$ a (real-valued) $C_0^\infty(E(1))$-function $\varphi_t \colon E(1) \to [0,1]$ with the properties (see e.g. [ST, p. 171])

(a) $\varphi_t(x) = 1$ for $x \in E$,

(b) $\varphi_t(x) = 0$ for $x \in E(1) \setminus E(t)$,

(c) $|\nabla \varphi_t(x)| \le b_1/t$.

The function $v(x) = u(x)\varphi_t(x)$ is admissible for the definition of $\mathrm{cap}(E(t),E)$ (see 7.2), and hence

$$\mathsf{M}\big(\Delta(\partial E(t), E)\big) = \mathrm{cap}(E(t),E) \le \int_{\mathbf{R}^n} |\nabla v|^n \, dm \, .$$

Since $|\nabla v(x)|^n \le 2^n\big(|\nabla u(x)|^n \varphi_t(x)^n + |\nabla \varphi_t(x)|^n |u(x)|^n\big)$ we get by the properties (a) and (c) of φ_t

$$\mathrm{cap}(E(t),E) \le 2^n \int_{\mathbf{R}^n} |\nabla u|^n \, dm + (2b_1)^n t^{-n} \int_{\mathbf{R}^n} |u(x)|^n \, dm \, .$$

Moreover, by Lemma 6.26 we obtain for $t \in (0,1]$

$$\int_{\mathbf{R}^n} |u(x)|^n \, dm \le 2^n (R+1)^n \int_{\mathbf{R}^n} |\nabla u(x)|^n \, dm \, .$$

Hence

$$\mathrm{cap}(E(t),E) \le a(t) \int_{\mathbf{R}^n} |\nabla u|^n \, dm \le a(t)\big(\epsilon + \mathrm{cap}(E(1),E)\big) \, ,$$

$$a(t) = 2^n (1 + 2^n b_1^n (R+1)^n t^{-n})$$

for $t \in (0,1]$. For $t \ge 1$ we define $a(t) = a(1)$. Because $\epsilon > 0$ is arbitrary the proof follows from this last inequality in view of 7.8 and 5.3. ❐

We already know by Exercise 6.25 that the inequality $a(t) \le a_1 t^{-n}$ of Lemma 6.27 provides the best possible integer power for the growth of $a(t)$. Next, we shall show that this rate t^{-n} of growth for $a(t)$ is in fact attained.

6.28. Example. We shall show that there exists a constant $b_1 > 0$ and for arbitrarily small $t \in (0,1)$ a set $E = E_t$ in $\mathbf{R}^n$ such that

(6.29) $$M(E,t) = \mathsf{M}\big(\Delta(E, \partial E(t))\big) \ge b_1 t^{-n}$$

where $E(t) = E + B^n(t)$.

Let $Q = [0,1]^{n-1} \times \{0\}$ and let $s \in (0,1)$. It follows from 5.11 that

(6.30) $$M(Q,s) \ge s^{1-n} \, .$$

Fix $k \geq 4$ and let $Q_j = Q + 2^{-k} j e_n$, $j = 0, \ldots, 2^k$. Set $E_k = \bigcup_{j=0}^{2^k} Q_j$. For $t \in (2^{-k-2}, 2^{-k-1})$ we obtain by 5.4 and (6.30)

$$M(E_k, t) \geq (2^k + 1) t^{1-n} \geq \tfrac{1}{4} t^{-n} .$$

In conclusion, we have proved (6.29) with $b_1 = \frac{1}{4}$.

6.31. Remarks. Modulus estimates in the spherical metric appear in [LV2, I.6.5], [V7, Section 12], [SR1], [MRV2, Lemma 3.11], and in [N2]. This section is taken from [VU8]. H. Renggli [REN] and W. P. Ziemer [Z2] have also constructed some set functions related to moduli of curve families.

7. The capacity of a condenser

In the present section we shall introduce, as a special case of curve families and their moduli, the notion of a condenser and its capacity, and we shall examine various properties of condensers. An important property of the capacity of a condenser is that it decreases under a special geometric transformation called symmetrization. Of the several kinds of symmetrization discussed in the literature (see e.g. [PS], [G1], [S1], [R12, p. 74]) we shall consider only spherical symmetrization. An immediate consequence of the above-mentioned monotoneity is the fact that condensers obtained as a result of spherical symmetrization are of extremal character — their capacities yield lower bounds for the capacities of a wide class of condensers in $\mathbf{R}^n$. The extremal condensers of Grötzsch and Teichmüller are of particular importance, and the well-known estimates for the capacities of these condensers are given in this section.

One of the main themes of this section is the relationship of the capacity of a condenser to its geometric structure. The hyperbolic and quasihyperbolic geometries are useful instruments in the study of this interrelation in Sections 7 and 8. In this context the hyperbolic and quasihyperbolic geometries are useful for proving estimates for the capacity only of ring domains with non-degenerate complementary components.

7.1. Definition. For $j = 1, \ldots, n$ let $R_j^n = \{ x \in \mathbf{R}^n : x_j = 0 \}$ and let $T_j \colon \mathbf{R}^n \to R_j^n$ be the orthogonal projection $T_j x = x - x_j e_j$. Let $D \subset \mathbf{R}^n$ be an open set and $u \colon D \to \mathbf{R}$ a continuous function. The function u is called *absolutely continuous on lines*, abbreviated as ACL, if for every cube Q with $\overline{Q} \subset D$, the set $A_j \subset T_j D \subset R^n_j$ of all points $z \in T_j Q$ such that the function $t \mapsto u(z + te_j)$, $z + te_j \in Q$, is not absolutely continuous as a function of a single variable [HS, p. 282], satisfies $m_{n-1}(A_j) = 0$ for all $j = 1, \ldots, n$.

By well-known properties of absolutely continuous functions of a single variable the derivative exists almost everywhere and is Borel-measurable (see [HS, p. 285], [V7, pp. 87–89]. From this fact and from Fubini's theorem it follows that an ACL function $u: D \to \mathbf{R}$ has partial derivatives with respect to every variable $x_1, \ldots, x_n$ a.e. (with respect to n-dimensional Lebesgue measure) in D. We say that an ACL function $u: D \to \mathbf{R}$ is ACL^p, $p \geq 1$, if $\partial u(x)/\partial x_j \in L^p(K)$, $j = 1, \ldots, n$, whenever $K \subset D$ is compact. A vector-valued function is said to be ACL (ACL^p) if and only if each coordinate function is in this class.

7.2. Definition. Let $A \subset \mathbf{R}^n$ be open and let $C \subset A$ be compact. The pair $E = (A, C)$ is called a *condenser*. Its *p-capacity* is defined by

$$p\text{-cap}\, E = \inf_u \int_{\mathbf{R}^n} |\nabla u|^p dm \,, \tag{7.3}$$

where the infimum is taken over the family of all non-negative ACL^p functions u with compact support in A such that $u(x) \geq 1$ for $x \in C$. Here

$$\nabla u(x) = \left(\frac{\partial u}{\partial x_1}(x), \ldots, \frac{\partial u}{\partial x_n}(x) \right) .$$

A function u with these properties is called an *admissible function.*

It follows from (7.3) that p-cap E is invariant under translations and orthogonal maps.

Without alteration of the real number p-cap E, one can take the infimum in (7.3) over several other classes of functions as can be shown by approximation. For instance one may take functions $u \in C^\infty(A)$ with compact support in A and $u(x) \geq 1$ for $x \in C$ (see [MRV1]). The following *monotone property* of condensers is a consequence of the definition. If (A, C) and (A', C') are condensers with $A' \subset A$ and $C \subset C'$, then

$$p\text{-cap}\,(A', C') \geq p\text{-cap}\,(A, C) \,. \tag{7.4}$$

The p-capacity of (A, C) reflects the metric structure of the pair C, $\mathbf{R}^n \setminus A$ as we shall see later on. If $p = n$ we denote n-cap (A, C) simply by cap(A, C) and call it the *capacity* or *conformal capacity* of the condenser (A, C).

An ACL^p function $u\colon D \to \mathbf{R}^m$, where $D \subset \mathbf{R}^n$ is open, is said to be *absolutely continuous* on the rectifiable curve $\alpha\colon [a,b] \to D$ iff $f \circ \alpha^0\colon [0, \ell(\alpha)] \to \mathbf{R}^m$ is absolutely continuous as a function of one variable.

We shall make use of the following result of B. Fuglede [F], [V7, 28.1, 28.2].

7.5. Lemma. *Let D be an open set in $\mathbf{R}^n$ and let $f\colon D \to \mathbf{R}^m$ be ACL^p. Then the family of all locally rectifiable paths in D having a closed subpath on which f is not absolutely continuous, is p-exceptional.*

7.6. Lemma. *Let G be a domain in $\mathbf{R}^n$, let $u\colon G \to \mathbf{R}$ be an ACL^p function, $-\infty < a < b < \infty$, and let $A, B \subset G$ be non-empty sets such that $u(x) \le a$ for $x \in A$ and $u(x) \ge b$ for $x \in B$. Then*

$$\mathsf{M}_p\big(\Delta(A,B;G)\big) \le (b-a)^{-p} \int_G |\nabla u|^p dm .$$

Proof. Define an ACL^p function $v\colon G \to \mathbf{R}$ by

$$v(x) = \frac{u(x) - a}{b - a}\,, \quad x \in G .$$

Then $v(y) \ge 1$ for $y \in B$ and $v(y) \le 0$ for $y \in A$. Let $\Delta = \{\gamma \in \Delta(A,B;G) :$ γ is rectifiable$\}$ and

$$\Delta_u = \{\, \gamma \in \Delta : v \text{ is not absolutely continuous on a closed subpath of } \gamma \,\} .$$

Fix $\gamma \in \Delta \setminus \Delta_u$ with the normal representation $\gamma^0\colon [0,c] \to G$, $c = \ell(\gamma)$, and with $\gamma^0(0) \in A$, $\gamma^0(c) \in B$. Then γ^0 has a Lipschitz constant 1 and $\big|(\gamma^0)'(t)\big| = 1$ a.e. in $[0,c]$ (see [V7, 2.4]). By [V7, 1.3] we get

$$\begin{aligned} 1 &\le v(\gamma^0(c)) - v(\gamma^0(0)) \le \int_0^c \big|(v \circ \gamma^0)'(t)\big|\, dt \\ &\le \int_0^c \big|(\nabla v \circ \gamma^0)(t)\big|\,\big|(\gamma^0)'(t)\big|\, dt = \int_\gamma |\nabla v|\, ds . \end{aligned} \tag{7.7}$$

Since v is ACL^p, $|\nabla v|$ is a Borel function and thus $|\nabla v| \in \mathcal{F}(\Delta \setminus \Delta_u)$ in view of (7.7). By 7.5 we obtain

$$\mathsf{M}_p\big(\Delta(A,B;G)\big) \le \int_G |\nabla v|^p dm = (b-a)^{-p} \int_G |\nabla u|^p dm . \quad \square$$

Let (A,C) be a condenser for which A is bounded. It follows from 7.6 that

$$\mathsf{M}_p(\Delta(C,\partial A;A)) \le \int_A |\nabla u|^p dm$$

for every u in ACL^p with $C \subset \{z \in A : u(z) \ge 1\}$ and $\partial A \subset \{z \in \mathbf{R}^n : u(z) \le 0\}$. Thus

$$\mathsf{M}_p(\Delta(C,\partial A;A)) \le p\text{-cap}\,(A,C)\,.$$

Also the converse inequality holds true according to the following result of W. P. Ziemer [Z1], but the proof is longer and will be omitted.

7.8. Theorem. *If $E = (A,C)$ is a bounded condenser in $\mathbf{R}^n$, then*

$$p\text{-cap}\,E = \mathsf{M}_p(\Delta(C,\partial A;A))\,.$$

7.9. Remark. By 5.9 the curve family on the right side of Theorem 7.8 may be replaced by some other families as well. We shall need 7.8 mainly in the case $p = n$. We now show that 7.8 holds also for unbounded condensers if $p = n$. Let (A,C) be an unbounded condenser, let $z \in C$ and $r > 1 + d(C)$, and $A_r = A \cap B^n(z,r^2)$. By the monotone property of the capacity, by 7.8, 5.9, and 5.14 we obtain

$$\begin{aligned}\mathrm{cap}(A,C) \le \mathrm{cap}(A_r,C) &= \mathsf{M}(\Delta(C,\partial A_r;A_r)) \\ &\le \mathsf{M}(\Delta(C,\partial A;A)) + \mathsf{M}(\Delta(C,S^{n-1}(z,r^2))) \\ &\le \mathsf{M}(\Delta(C,\partial A;A)) + \omega_{n-1}(\log r)^{1-n}\end{aligned}$$

Letting $r \to \infty$ shows that $\mathrm{cap}(A,C) \le \mathsf{M}(\Delta(C,\partial A;A))$. The converse inequality follows from 7.6 and 7.7. In conclusion, we have proved that the equality

$$\mathrm{cap}(A,C) = \mathsf{M}(\Delta(C,\partial A;A)) \tag{7.10}$$

holds whenever (A,C) is a condenser in $\mathbf{R}^n$, whether A is bounded or not.

We now extend the definition of a condenser to $\overline{\mathbf{R}}^n$. Assume that $C \subset \overline{\mathbf{R}}^n$ is compact and that there exists an open set $A \subset \overline{\mathbf{R}}^n$ with $A \ne \overline{\mathbf{R}}^n$ and $C \subset A$. Then we say that (A,C) is a condenser in $\overline{\mathbf{R}}^n$ and define its (n-)capacity by (7.10). In view of 7.8 this extended definition is compatible with the definition (7.3) in case $A \subset \mathbf{R}^n$. (We shall not need the p-capacity, $p \ne n$, of a condenser in $\overline{\mathbf{R}}^n$.)

Now let (A, E) be a condenser in $\mathbf{R}^n$ or $\overline{\mathbf{R}}^n$. It follows from (7.10) and the conformal invariance of the modulus 5.17 that $\operatorname{cap}(A, E)$ is a conformal invariant. Likewise, by virtue of (7.10), many properties of $\operatorname{cap}(A, E)$ may be derived directly from the properties of the modulus. In particular, we shall often make use of Remark 5.9 specialized to condensers. If $p \neq n$, then $p-\operatorname{cap}(A, E)$ is not invariant under conformal mappings, while $n-\operatorname{cap}(A, E)$ has this invariance property. The corresponding property of the modulus immediately yields this conclusion.

7.11. Lemma. *Let* (A, F) *be a condenser with* $A \subset \mathbf{B}^n$. *Then there are positive numbers* a_1 *depending only on* n, *and* a_2 *depending only on* n *and* $d(F, \partial A)$ *such that*

$$a_1 c(F) \leq \operatorname{cap}(A, F) \leq a_2 c(F) \, .$$

Proof. By the proof of 6.19 and 7.8

$$\operatorname{cap}(A, F) \geq d_4 \min\{\, c(\overline{\mathbf{R}}^n \setminus A,\, 0)\, ,\, c(F, 0)\, \} \, .$$

Since $A \subset \mathbf{B}^n$,

$$c(\overline{\mathbf{R}}^n \setminus A,\, 0) = \omega_{n-1} (\log \sqrt{3}\,)^{1-n} = c(\overline{\mathbf{R}}^n, 0) \geq c(F, 0)$$

(cf. (6.10), (6.12)) and thus we obtain the desired lower bound

$$\operatorname{cap}(A, F) \geq d_4 c(F) \, .$$

For the upper bound let $t = q(F, \mathbf{R}^n \setminus A)$. Then by 1.17 $t \geq \frac{1}{2} d(F, \mathbf{R}^n \setminus A)$ since $F \subset A \subset \mathbf{B}^n$. By the proof of 6.20 and 6.14

$$\begin{aligned} \operatorname{cap}(A, F) &\leq 4\, b(t/8) \min\{\, c(F, 0)\, ,\, c(\mathbf{R}^n \setminus A,\, 0)\, \} \\ &\leq 4\, d_1\, b(d(F, \partial A)/16)\, c(F) \, , \end{aligned}$$

which yields the desired upper bound. (Note that a slightly better upper bound can be derived from 6.27.) ❐

7.12. Definition. A compact set E in $\mathbf{R}^n$ is said to be of *capacity zero*, denoted $\operatorname{cap} E = 0$, if there exists a bounded open set A with $E \subset A$ and $\operatorname{cap}(A, E) = 0$. A compact set $E \subset \overline{\mathbf{R}}^n$, $E \neq \overline{\mathbf{R}}^n$, is said to be of *capacity zero* if E can be mapped by a Möbius transformation onto a bounded set of capacity zero. Otherwise E is said to be of positive capacity, and we write $\operatorname{cap} E > 0$.

7.13. Remarks. By conformal invariance the second part of the above definition is independent of the choice of Möbius transformation. We next show that the first part of the definition is independent of the choice of bounded open set A with $A \supset E$. Indeed, if $A_j \supset E$, $j = 1,2$, are both bounded, say $A_j \subset B^n(R)$, $j = 1,2$, then by 6.1(4)

$$d_2 = c(\overline{\mathbf{R}}^n) \geq c(\overline{\mathbf{R}}^n \setminus A_j) \geq d_3/\sqrt{2+R^2}\,; \quad j = 1,2\,.$$

This inequality together with 6.1(5),(6) yields for $j = 1,2$

$$d_4 \min\{\, c(E),\ d_3/\sqrt{2+R^2}\,\} \leq \operatorname{cap}(A_j, E) \leq d_5 \min\{\, c(E),\ d_2\,\}$$

where d_2, d_3, d_4 are positive numbers depending only on n and where d_5 depends also on $q\,(E,\ (\partial A_1) \cup (\partial A_2))$. In other words, $\operatorname{cap}(A_j, E) = 0$ if and only if $c(E) = 0$. Hence the condition $\operatorname{cap} E = 0$ is independent of the choice of open bounded set A with $E \subset A$ (see also [R2, Lemma 2]). This argument also shows that in the above definition of $\operatorname{cap} E = 0$, $E \subset \mathbf{R}^n$ compact, one can replace the bounded set A by a ball $B^n(r)$ with $r \geq d(0,E) + 2d(E)$, say.

It should be observed that we have only defined the conditions $\operatorname{cap} E = 0$ and $\operatorname{cap} E > 0$ for a compact set E and that in the latter case the "capacity" of E will not be specified as a real number. In view of 7.9 and (5.15) countable compact sets are examples of sets of capacity zero. The following theorem shows that sets of capacity zero are always very thin [R12, p. 72]. The definition of the Hausdorff dimension and the α-dimensional Hausdorff measure can be found in [FA] and [MAT2].

7.14. Lemma. *Suppose that F is a compact set in $\mathbf{R}^n$ of capacity zero. Then for every $\alpha > 0$, the α-dimensional Hausdorff measure $\Lambda_\alpha(F)$ of F is zero. In particular,* $\operatorname{int} F = \emptyset$, *and F is totally disconnected.*

7.15. Remarks. (1) In the dimension $n = 2$ the logarithmic capacity is often used in complex analysis. H. Wallin [W1] has proved that a compact set is of logarithmic capacity zero if and only if it is of capacity zero in the sense of the above definition ($n = 2$). He has also constructed a compact Cantor-type set E in $\mathbf{R}^n$ of positive capacity (in the sense of 7.12) with $\Lambda_\alpha(E) = 0$ for all $\alpha > 0$. See also V. G. Maz'ya–V. P. Khavin [MK].

(2) Various sufficient or necessary conditions for capacity zero can be found in the literature [MK], [W2], [R12, p. 71], [R7], [MV], [V9].

7.16. The spherical symmetrization. If $x_0 \in \mathbf{R}^n$, $E \subset \overline{\mathbf{R}}^n$ and if L is a ray from x_0 to ∞, then the *spherical symmetrization* E^* of E in L is defined as follows:

(1) $x_0 \in E^*$ iff $x_0 \in E$,

(2) $\infty \in E^*$ iff $\infty \in E$,

(3) for $r \in (0,\infty)$, $E^* \cap S^{n-1}(x_0,r) \neq \emptyset$ iff $E \cap S^{n-1}(x_0,r) \neq \emptyset$, in which case $E^* \cap S^{n-1}(x_0,r)$ is a closed spherical cap centered on L with the same m_{n-1} measure as $E \cap S^{n-1}(x_0,r)$.

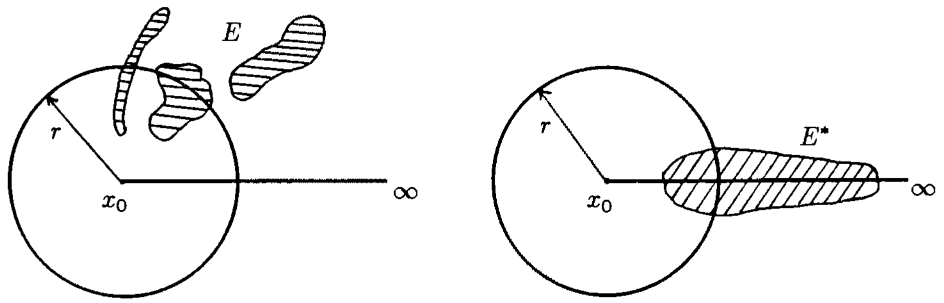


Diagram 7.1.

Let (A,C) be a condenser and $x_0 \in \mathbf{R}^n$. Denote by C^* and B the spherical symmetrizations of C and $\mathbf{R}^n \setminus A$ in two opposite rays L_1 and L_2 emanating from x_0, and let $A^* = \mathbf{R}^n \setminus B$. Then it is easy to verify that (A^*, C^*) is a condenser [S1]. An important property of spherical symmetrization is given in the following theorem [G1], [S1].

7.17. Theorem. *If* (A,C) *is a condenser, then for* $p \geq 1$

$$p\text{-cap}\,(A,C) \geq p\text{-cap}\,(A^*,C^*)\,.$$

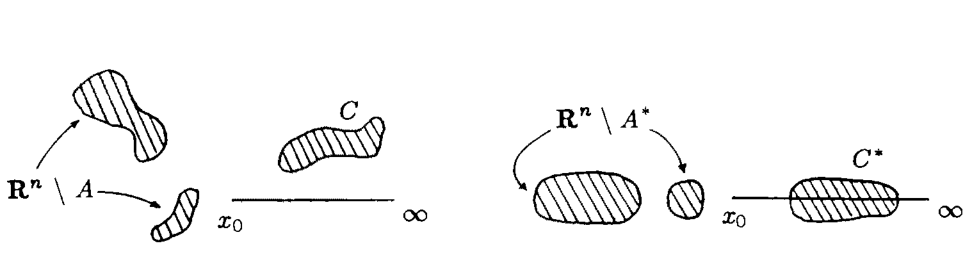


Diagram 7.2.

This inequality is sharp in the sense that there is equality if $(A^*, C^*) = (A, C)$ (e.g. $x_0 = 0$, $C = [0, e_1]$, $A = B^n(2)$ and L_1 is the positive x_1-axis). Note that the minorant p-$\operatorname{cap}(A^*, C^*)$ in 7.17 depends on the choice of the center of symmetrization, the point x_0, in an essential way. For instance, if $n \geq 3$, $E_j = \{x \in S^{n-1}(2^{-j}) : x_3 = 0\}$, $E = \{0\} \cup \left(\bigcup_{j=1}^{\infty} E_j\right)$ and if E^* is the spherical symmetrization of E in the positive x_1-axis (in which case $x_0 = 0$), then $E^* = \{0, \frac{1}{2}e_1, \frac{1}{4}e_1, \ldots\}$ and clearly $\operatorname{cap}(\mathbf{B}^n, E^*) = 0$. It is left as an exercise for the reader to find a spherical symmetrization with center $\neq 0$ which provides a strictly positive minorant for $\operatorname{cap}(\mathbf{B}^n, E)$.

7.18. The Grötzsch and Teichmüller rings. Let us recall the Grötzsch and Teichmüller rings $R_{G,n}(s)$ and $R_{T,n}(s)$ which were introduced in Section 5. They can also be understood as condensers in the following way:

$$R_{G,n}(s) = \left(\mathbf{R}^n \setminus \{te_1 : t \geq s\}, \overline{\mathbf{B}}^n\right), \; s \in (1, \infty),$$

$$R_{T,n}(s) = \left(\mathbf{R}^n \setminus \{te_1 : t \geq s\}, [-e_1, 0]\right), \; s \in (0, \infty).$$

We define functions $\Phi = \Phi_n$ and $\Psi = \Psi_n$ by $\operatorname{mod} R_{G,n}(s) = \log \Phi(s)$ and $\operatorname{mod} R_{T,n}(s) = \log \Psi(s)$. In other words (cf. (5.52))

$$\text{(7.19)} \qquad \begin{cases} \operatorname{cap} R_{G,n}(s) = \omega_{n-1} (\log \Phi(s))^{1-n} = \gamma_n(s), \\ \operatorname{cap} R_{T,n}(s) = \omega_{n-1} (\log \Psi(s))^{1-n} = \tau_n(s). \end{cases}$$

7.20. Lemma. *The function $\Phi(t)/t$ is increasing for $t > 1$ and $\Psi(t-1) = \Phi(\sqrt{t})^2$ for $t > 1$. Moreover, the functions γ_n and τ_n are strictly decreasing.*

Proof. For the first part fix $1 < s < t$, let $R = R_{G,n}(t)$ and let R' and R'' be the two rings into which R is split by the sphere $|x| = t/s$. By 5.24 and 5.14 we obtain

$$\begin{aligned} \log \Phi(t) &= \operatorname{mod} R \geq \operatorname{mod} R' + \operatorname{mod} R'' \\ &= \log(t/s) + \log \Phi(s) \end{aligned}$$

whence $\Phi(t)/t \geq \Phi(s)/s$ as desired. It follows, in particular, that Φ and Ψ are strictly increasing and hence by (7.19) γ_n and τ_n are strictly decreasing. The asserted identity is the functional identity 5.53 rewritten. ☐

By 7.20 the function $\log \Phi(t) - \log t$ is increasing and therefore has a limit as $t \to \infty$. We define a number λ_n by

$$\text{(7.21)} \qquad \log \lambda_n = \lim_{t \to \infty} \left(\log \Phi(t) - \log t\right).$$

This number is sometimes called the Grötzsch (ring) constant. Only for $n = 2$ is the exact value of the Grötzsch constant known, $\lambda_2 = 4$ [LV2, p. 61, (2.10)]. Various estimates for λ_n, $n \geq 3$, are given in [G1, p. 518], [C1, pp. 239–241], [AN2]. For instance it is known that $\lambda_n \in [4, 2e^{n-1})$,

$$\lambda_n \leq 4 \exp\left(\int_1^\infty \frac{a(n,s)}{s} \, ds \right) \; ; \; a(n,s) = \left(\frac{s^2+1}{s^2-1} \right)^{\frac{n-2}{n-1}} - 1 \, ,$$

and that $\lambda_n^{1/n} \to e$ as $n \to \infty$. These technical results will not be proved here. Some of them are summarized in the next lemma.

7.22. Lemma. *For each $n \geq 2$ there exists a number $\lambda_n \in [4, 2e^{n-1})$, $\lambda_2 = 4$, such that*

(1) $t \leq \Phi(t) \leq \lambda_n t \, , \; t > 1 \, ,$

(2) $t + 1 \leq \Psi(t) \leq \lambda_n^2 \, (t+1) \, , \; t > 1 \, .$

Furthermore, $\lambda_n^{1/n} \to e$ as $n \to \infty$ and, in particular, $\lambda_n \to \infty$ as $n \to \infty$.

Proof. The bounds $4 \leq \lambda_n \leq 2e^{n-1}$ are given in [G1], [AN2]. The lower bound in (1) follows from the fact that the boundary components of $R_{G,n}(t)$ are separated by the annulus $A = B^n(t) \setminus \overline{\mathbf{B}}^n$ with $\operatorname{mod} A = \log t$ and from 5.3. The upper bound in (1) follows from 7.20 and the definition (7.21) of λ_n above. Inequality (2) follows from (1) and 7.20, and the last assertion is proved in [AN2]. $\square$

Because of the functional identity 5.53 the properties of τ can be derived from those of γ and conversely. A simple argument similar to 5.63(3) shows that the strictly decreasing function τ is continuous on $(0, \infty)$. In what follows we may use these simple properties without notice.

The following fundamental difference between dimensions $n = 2$ and $n \geq 3$ should be observed: for $n \geq 3$ no explicit expression like (5.56) is known for $\gamma_n(s)$. It is an interesting open problem to find such a formula also for the multidimensional case.

The Grötzsch and Teichmüller condensers have some important extremal properties which are connected with the spherical symmetrization. In what follows we shall often require a lower bound for the capacity of a ring domain in terms of the Teichmüller capacity $\tau_n(s)$ which follows from the spherical symmetrization lemma 7.17. For this reason various estimates for $\gamma_n(s)$ and $\tau_n(s)$ will be very useful —

in fact they will be necessary for our later work in the multidimensional case $n \ge 3$ when no exact formulae for $\tau_n(s)$ or $\gamma_n(s)$ are known.

Before giving these estimates we shall discuss qualitatively the behavior of $\tau_n(s)$ and $\gamma_n(s)$. First we note that by (5.14) and 5.32 the limit values of $\gamma_n(s)$ and $\tau_n(s)$ are

$$\begin{cases} \lim_{s\to 1+} \gamma_n(s) = \infty \,, \; \lim_{s\to\infty} \gamma_n(s) = 0 \,, \\ \lim_{s\to 0+} \tau_n(s) = \infty \,, \; \lim_{s\to\infty} \tau_n(s) = 0 \,. \end{cases} \tag{7.23}$$

For convenience we set $\gamma_n(\infty) = 0 = \tau_n(\infty)$ and $\gamma_n(1) = \infty = \tau_n(0)$.

Lemma 7.22 yields the inequalities

$$\begin{cases} \omega_{n-1} \left(\log \lambda_n s\right)^{1-n} \le \gamma_n(s) \le \omega_{n-1}(\log s)^{1-n} \,, \\ \omega_{n-1} \left(\log(\lambda_n^2 s)\right)^{1-n} \le \tau_n(s-1) \le \omega_{n-1}(\log s)^{1-n} \end{cases} \tag{7.24}$$

for $s > 1$. In passing we shall show how this upper bound for $\gamma_n(s)$ can be slightly improved. First, fix $s > 1$ and choose $h \in \mathcal{GM}(\mathbf{B}^n)$ with $h[0, \frac{1}{s}e_1] = [-ae_1, ae_1]$, $a > 0$. Then $a = s - \sqrt{s^2-1}$ by 2.42 and by conformal invariance 5.17, 5.3, and 5.14

$$\begin{aligned} \gamma_n(s) &= \mathsf{M}\left(\Delta(S^{n-1}, [0, \tfrac{1}{s}e_1])\right) = \mathsf{M}\left(\Delta(S^{n-1}, [-ae_1, ae_1])\right) \\ &\le \omega_{n-1}\left(\log(s + \sqrt{s^2-1})\right)^{1-n} < \omega_{n-1}(\log s)^{1-n} \,. \end{aligned} \tag{7.25}$$

We note that (7.25) yields a slightly better upper bound for $\gamma_n(s)$ than (7.24). Note also that by combining the first inequality in (7.24) with (7.25) and letting $s \to \infty$ yields $\lambda_n \ge 2$ for all $n \ge 2$. An even better upper bound for $\gamma_n(s)$ will be given in Lemma 7.26.

Each of the bounds for $\gamma_n(s)$ in (7.24) is asymptotically sharp as s tends to ∞, but not of the correct order as s tends to 1, as can be seen from 7.26 below. The following theorem due to G. D. Anderson [AN1] yields inequalities that are asymptotically sharp as s tends to 1.

7.26. Theorem. *For $s \in (1,\infty)$ and $n \ge 2$*

(1) $\gamma_n(s) \le \omega_{n-1}\mu(1/s)^{1-n} < \omega_{n-1}\left(\log(s + 3\sqrt{s^2-1})\right)^{1-n}$,

(2) $2^{n-1}c_n \log\left(\frac{s+1}{s-1}\right) \le \gamma_n(s) \le 2^{n-1}c_n\, \mu\left(\frac{s-1}{s+1}\right) < 2^{n-1}c_n \log\left(4\,\frac{s+1}{s-1}\right)$.

Moreover, if $s \in (0,\infty)$ and $a = 1 + 2(1+\sqrt{1+s})/s$, then

(3) $c_n \log a \le \tau_n(s) \le c_n \mu(1/a) < c_n \log(4a)$

and $(1+1/\sqrt{s})^2 \le a \le (1+2/\sqrt{s})^2$ *hold true. Furthermore, when* $n=2$, *the first inequality in* (1), *the second inequality in* (2), *and the second inequality in* (3) *hold as identities.*

Proof. (1) The proof of the inequality $\gamma_n(s) \le \omega_{n-1}\mu(1/s)^{1-n}$ will be omitted (see [AN1]). The second inequality follows from (5.58).

(2) & (3) It is left as an easy exercise for the reader to verify that (2) and (3) are equivalent, that is, one can be derived from the other in view of (5.53). Hence it suffices to prove (2). Here we shall only prove the lower bound in (2); the proof of the upper bound will be omitted (see [AN2]). Let h be an inversion in $S^{n-1}(-e_n, \sqrt{2})$ which maps $\mathbf{B}^n$ onto $\mathbf{H}^n$ and 0 to e_n. By (2.22) h preserves hyperbolic distances. Because

$$\gamma_n(s) = \mathsf{M}\left(\Delta([0, \tfrac{1}{s}e_n], S^{n-1})\right)$$

and $\rho_{\mathbf{B}^n}(0, \frac{1}{s}e_n) = \log \frac{s+1}{s-1}$ we see by (2.6) and (2.17) that $h(\frac{1}{s}e_n) = \frac{s-1}{s+1}e_n$. By conformal invariance, 5.26, and 5.32 we obtain

$$\begin{aligned}\gamma_n(s) &= \mathsf{M}\big(\Delta\big([\tfrac{s-1}{s+1}e_n, e_n], \partial\mathbf{H}^n\big)\big)\\ &= 2^{n-1}\mathsf{M}\big(\Delta\big([\tfrac{s-1}{s+1}e_n, e_n], [-\tfrac{s-1}{s+1}e_n, -e_n]\big)\big)\\ &\ge 2^{n-1}c_n \log\frac{s+1}{s-1}\,,\end{aligned}$$

which is the desired lower bound. The proof for the bounds for a is elementary.

The assertions concerning the case $n=2$ follow from (5.56), (5.57), (5.59), 5.60(1), and 5.30. ❒

It should be observed that 7.26(1) yields a slightly better explicit bound than (7.25).

We shall next summarize the preceding inequalities for $\gamma_n(s)$. Let

$$\begin{aligned}u_1(s) &= \omega_{n-1}\mu(1/s)^{1-n}\,, \quad u_2(s) = 2^{n-1}c_n\mu\Big(\frac{s-1}{s+1}\Big)\,,\\ v_1(s) &= \omega_{n-1}(\log\lambda_n s)^{1-n}\,,\ v_2(s) = 2^{n-1}c_n\log\frac{s+1}{s-1}\,.\end{aligned}$$

By (7.24) and 7.26

$$\max\{v_1(s), v_2(s)\} \le \gamma_n(s) \le \min\{u_1(s), u_2(s)\}\,. \tag{7.27}$$

For $n = 2$ the right side of (7.27) is sharp. In fact, it follows from (5.56) (or (5.59)) that for $n = 2$ the right side holds as equality for all $s > 1$. One can rewrite (7.27) for $\tau_n(s)$ using the functional identity 5.53.

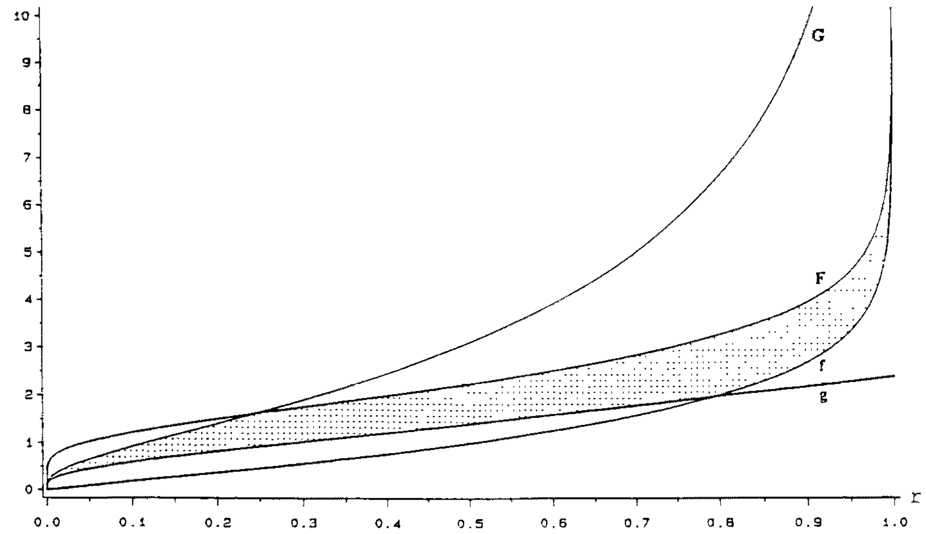


Diagram 7.3. The graph of $\gamma_3(1/r)$, $0 < r < 1$, lies in the shaded region.

Lower bounds: $f(r) = 4c_3 \log \dfrac{1+r}{1-r}$, $g(r) = \dfrac{4\,\pi}{\log^2(9.9002/r)}$ $(\lambda_3 < 9.9002)$.

Upper bounds: $F(r) = 4c_3\mu\Big(\dfrac{1-r}{1+r}\Big)$, $G(r) = \dfrac{4\,\pi}{\big(\mu(r)\big)^2}$ (from [AVV3]).

7.28. Remarks. (1) The last inequalities in 7.26(2) and (3) can be improved in view of (5.67).

(2) The inequality 7.26(3) can also be written as follows

$$c_n \log(1+t) \le \tau\Big(\frac{4(1+t)}{t^2}\Big) \le c_n\,\mu\Big(\frac{1}{1+t}\Big)\,,\quad t > 0\,.$$

7.29. Hyperbolic metric and capacity. As in Section 2 we let $J[x,y]$ denote the geodesic segment of the hyperbolic metric joining x to y, $x, y \in \mathbf{B}^n$. It is clear by conformal invariance that

$$\operatorname{cap}(\mathbf{B}^n, J[x,y]) = \operatorname{cap}(\mathbf{B}^n, T_x J[x,y])$$

where T_x is as defined in 1.34. We obtain by (2.25) and (7.25)

$$\text{(7.30)} \qquad \operatorname{cap}(\mathbf{B}^n, J[x,y]) = \gamma_n\Big(\frac{1}{\operatorname{th}\frac{1}{2}\rho(x,y)}\Big) \le \omega_{n-1}\big(-\log\operatorname{th}\tfrac{1}{4}\rho(x,y)\big)^{1-n} .$$

Next by (7.30), 7.26(2), and (5.58) we get

$$\text{(7.31)} \qquad \begin{aligned} 2^{n-1}c_n\,\rho(x,y) \le \operatorname{cap}(\mathbf{B}^n, J[x,y]) &\le 2^{n-1}c_n\,\mu(e^{-\rho(x,y)}) \\ &< 2^{n-1}c_n(\rho(x,y)+\log 4) . \end{aligned}$$

For large values of $\rho(x,y)$ (7.31) is quite accurate. For small $\rho(x,y)$ one obtains better inequalities than (7.31) by combining 7.26(1) and (7.30).

7.32. Lemma. *Let $x,y \in \mathbf{B}^n$ and let $E \subset \mathbf{B}^n$ be a continuum with $x,y \in E$. Then*

$$\operatorname{cap}(\mathbf{B}^n, E) \ge \operatorname{cap}(\mathbf{B}^n, J[x,y]) = \gamma_n\Big(\frac{1}{\operatorname{th}\frac{1}{2}\rho(x,y)}\Big) .$$

Proof. Let T_x be as in 1.34 and let $*$ denote spherical symmetrization in the positive x_1-axis. Then the center of symmetrization is the origin and by 7.17 we obtain

$$\operatorname{cap}(\mathbf{B}^n, E) = \operatorname{cap}(\mathbf{B}^n, T_x(E)) \ge \operatorname{cap}(\mathbf{B}^n, (T_x(E))^*) .$$

By (2.25) we see that $[0, (\operatorname{th}\frac{1}{2}\rho(x,y))e_1] \subset (T_x(E))^*$ and the proof follows from (7.30). $\square$

7.33. Exercise. Show that 5.34 follows from 7.32. [Hint: We may assume that $t=1$ in 5.34. Apply (7.31) and 2.41(1).]

The next result gives a very useful lower bound for the capacity of a ring domain.

7.34. Lemma. *Let $R = R(E,F)$ be a ring in $\mathbf{R}^n$ and let $a,b \in E$, $c,\infty \in F$ be distinct points. Then*

$$\operatorname{cap} R = \mathsf{M}(\Delta(E,F)) \ge \tau\Big(\frac{|a-c|}{|a-b|}\Big) .$$

Here equality holds for $E = [-e_1, 0]$, $a=0$, $b=-e_1$, $F = [se_1, \infty)$, $c = se_1$, $d=\infty$.

Proof. Observe first that the right side remains invariant under the similarity transformation $f(x) = (x-a)/|a-b|$. Then $|f(c)| = |a-c|/|a-b|$ and $f(a)=0$.

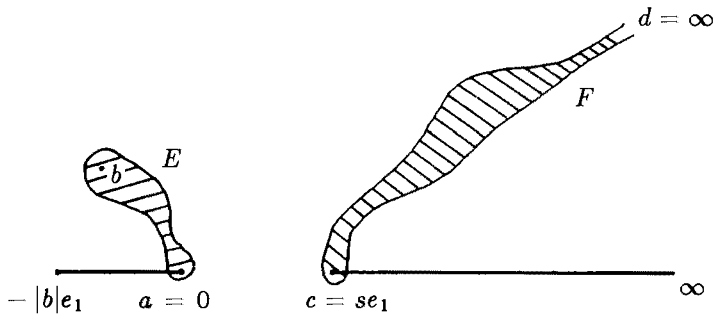


Diagram 7.4.

The spherical symmetrizations of fE and fF in the negative and positive x_1-axis, respectively, contain the complementary components of $R_{T,n}\big(\frac{|a-c|}{|a-b|}\big)$. Thus by 7.17

$$\operatorname{cap} R \geq \tau\Big(\frac{|a-c|}{|a-b|}\Big) . \quad \square$$

7.35. Lemma. *Let $R = R(E,F)$ be a ring in $\overline{\mathbf{R}}^n$ and let $a,b \in E$, $c,d \in F$ be distinct points. Then*

$$\operatorname{cap} R \geq \tau\,(|\,a,b,c,d\,|) .$$

Here equality holds if $b = s_1 e_1$, $a = s_2 e_1$, $c = s_3 e_1$, $d = s_4 e_1$, and $s_1 < s_2 < s_3 < s_4$.

Proof. By (1.29) we may assume that $a = 0$, $b = e_1$, $d = \infty$, and $|c| = |\,a,b,c,d\,|$. The proof follows now from 7.34. The assertion concerning the equality follows from 5.54(1). $\square$

7.36. Corollary. *Let $R = R(E,F)$ be a ring and let $a,b \in E$, $c,d \in F$ be distinct points in $\mathbf{R}^n$. Then*

$$\operatorname{cap} R \geq \tau\Big(\frac{t-s}{s(1-t)}\Big)$$

where

$$s = \frac{|a-b|}{|a-b|+|a-c|+|c-d|} , \quad t = \frac{|a-b|+|a-c|}{|a-b|+|a-c|+|c-d|} .$$

Here equality holds for $E = [0, se_1]$, $a = 0$, $b = se_1$, $F = [te_1, e_1]$, $c = te_1$, $d = e_1$, and $0 < s < t < 1$.

Proof. Because the points are finite, (1.15) and 7.35 yield

$$\operatorname{cap} R \geq \tau\Big(\frac{|a-c|\,|b-d|}{|a-b|\,|c-d|}\Big) .$$

The desired inequality follows from this and the fact that $|b-d| \leq |b-a|+|a-c|+|c-d|$. The statement concerning equality follows from 5.54(1). ❐

7.37. Corollary. *If* $R = R(E,F)$ *is a ring, then*

$$\text{(1)} \qquad \operatorname{cap} R \geq \tau\left(\frac{1}{q(E)q(F)}\right) ,$$

$$\text{(2)} \qquad \operatorname{cap} R \geq \tau\left(\frac{4q(E,F)}{q(E)q(F)}\right) .$$

Proof. (1) Choose $a, b \in E$ and $c, d \in F$ so that $q(a,b) = q(E)$ and $q(c,d) = q(F)$. Then

$$\frac{q(a,c)\,q(b,d)}{q(a,b)\,q(c,d)} \leq \frac{1}{q(E)\,q(F)}$$

and (1) follows from 7.35 because τ is decreasing.

(2) Choose $a \in E$, $c \in F$ such that $q(a,c) = q(E,F)$ and choose $b \in E$, $d \in F$ such that

$$q(a,b) \geq \tfrac{1}{2}\, q(E) \quad , \quad q(c,d) \geq \tfrac{1}{2}\, q(F) .$$

With this choice of a, b, c, d the proof follows from 7.35. ❐

7.38. Lemma. *Let* E *and* F *be disjoint continua in* $\mathbf{R}^n$ *with* $d(E), d(F) > 0$. *Then*

$$\mathsf{M}(\Delta(E,F)) \geq \tau(4m^2 + 4m) \geq c_n \log(1 + 1/m)$$

where $m = d(E,F)/\min\{d(E),\, d(F)\}$ *and* c_n *is as in* 5.32.

Proof. Fix $a \in E$, $c \in F$ with $|a-c| = d(E,F)$ and $b \in E$, $d \in F$ with $|a-b| = \frac{1}{2}\, d(E)$ and $|c-d| = \frac{1}{2}\, d(F)$, respectively. By 7.35 we obtain

$$\mathsf{M}(\Delta(E,F)) \geq \tau\Big(\frac{|a-c|\,|b-d|}{|a-b|\,|c-d|}\Big) \geq \tau\Big(\frac{|a-c|\,(|a-b|+|a-c|+|c-d|)}{|a-b|\,|c-d|}\Big) = \tau(u) .$$

Here

$$u = \frac{2\, d(E,F)\,\big(\, d(E) + 2d(E,F) + d(F)\big)}{d(E)\, d(F)} \leq 2m + 4m^2 + 2m ,$$

and the first inequality follows. The second one follows from 7.26(3). ❐

7.39. Corollary. *Let E and F be disjoint continua in $\mathbf{R}^n$ with $0 < d(E) \le d(F)$. Then*

$$\mathsf{M}(\Delta(E,F)) \ge 2^{1-n}\,\tau\Big(\frac{d(E,F)}{d(E)}\Big)\,.$$

Proof. The proof follows from 7.38 and 5.63. □

7.40. Exercise. (1) Show that if $R = R(E,F)$ is a ring in $\overline{\mathbf{R}}^n$, then

$$\operatorname{cap} R \ge q(E) c_n \log\Big(1 + \frac{q(F)}{q(E,F)}\Big)\,.$$

[Hint: Apply 7.37(2), 7.26(3), and (3.6).]

(2) Derive 5.42(2) from 7.37(1).

We are going to generalize the formula (7.31), which relates the hyperbolic distance $\rho(x,y)$ and the capacity of the condenser $(\mathbf{B}^n, J[x,y])$ in a simple fashion. Now we shall discuss instead of this particular condenser a general ring $R(E,F)$ and the hyperbolic distance will be replaced by the function

$$j_G(x,y) = \log\Big(1 + \frac{|x-y|}{\min\{d(x),\, d(y)\}}\Big)$$

which was introduced in (2.34).

7.41. Lemma. *If $R = R(E,F)$ is a ring with $\infty \notin E \cup F$, then*

$$\operatorname{cap} R \ge c_n \min\{\, j_{\mathbf{R}^n \setminus E}(F)\,,\, j_{\mathbf{R}^n \setminus F}(E)\,\}\,.$$

If $\infty \in F$, then

$$\operatorname{cap} R \ge c_n\, j_{\mathbf{R}^n \setminus F}(E)\,.$$

Proof. The proof follows immediately from 7.38, 7.34, and the definition of $j_G(A)$ (see (2.35) and 2.37). □

Applying this lemma with $E = S^{n-1}$, $F = J[x,y]$, $x,y \in \mathbf{B}^n$ we obtain in view of 2.41(1)

$$\begin{aligned}\operatorname{cap}(\mathbf{B}^n, J[x,y]) &\ge c_n\, j_{\mathbf{B}^n}(J[x,y]) = c_n\, j_{\mathbf{B}^n}(x,y)\\ &\ge \tfrac{1}{4} c_n\, \rho(x,y)\,.\end{aligned}$$

Hence 7.41 implies (7.31) with a slightly different constant. Thus we may regard 7.41 as a generalization of (7.31).

7.42. Remark. Lemma 7.41 has a converse which is valid even for disconnected sets E and F. Indeed one can show that for a given integer $n \geq 2$ there exists a homeomorphism $h_1\colon [0,\infty) \to [0,\infty)$ with the following properties. If E and F are compact disjoint sets in $\mathbf{R}^n$, then

$$\mathsf{M}(\Delta(E,F)) \leq h_1(T)\,,\quad T = \min\{\, j_{\mathbf{R}^n \setminus E}(F)\,, j_{\mathbf{R}^n \setminus F}(E) \,\}\,.$$

We shall outline a proof for this estimate. Clearly we may assume that $0 < d(E) \leq d(F)$. Set $t = d(E,F)$. Then 5.5 yields

$$\mathsf{M}(\Delta(E,F)) \leq \mathsf{M}(\Delta(E,\partial(E + B^n(t))) \leq \Omega_n\Big(\frac{d(E)+t}{t}\Big)^n\,,$$

while for $d(E) < t$ we obtain by (5.14)

$$\mathsf{M}(\Delta(E,F)) \leq \omega_{n-1}\Big(\log \frac{t}{d(E)}\Big)^{1-n}\,.$$

These two inequalities together imply the desired bound.

7.43. A special function. The functions γ_n and τ_n as well as their inverses and various combinations of these will occur often in Chapter III. Of particular importance is the function $\varphi_K\colon [0,1] \to [0,1]$, which will occur in the quasiregular version of the Schwarz lemma as well as in its many applications. This function is defined as follows. For $0 < r < 1$ and $K > 0$ we define a special function

$$\varphi_K(r) = \frac{1}{\gamma_n^{-1}(K\gamma_n(1/r))} = \varphi_{K,n}(r) \tag{7.44}$$

and set $\varphi_K(0) = 0$, $\varphi_K(1) = 1$. It is easy to see that $\varphi_K\colon [0,1] \to [0,1]$ is a homeomorphism. Next we shall derive some explicit estimates for φ_K. Recall first that by (7.24)

$$\omega_{n-1}(\log \lambda_n s)^{1-n} \leq \gamma_n(s) \leq \omega_{n-1}(\log s)^{1-n}$$

for $s > 1$. It is left as an exercise for the reader to derive from this the following inequality

$$t^\alpha/\lambda_n \leq \gamma_n^{-1}(K\gamma_n(t)) \leq \lambda_n^\alpha t^\alpha \tag{7.45}$$

for all $t > 1$ and $K > 0$, where $\alpha = K^{1/(1-n)}$. From (7.45) it follows that

$$r^\alpha \lambda_n^{-\alpha} \leq \varphi_K(r) \leq \lambda_n r^\alpha \tag{7.46}$$

holds for all $K > 0$ and $r \in (0,1)$.

It is easy to see that $0 < A \leq B < \infty$ implies $\varphi_A(r) \leq \varphi_B(r)$. In particular, $\varphi_{1/K}(r) \leq r = \varphi_1(r) \leq \varphi_K(r)$ for $K \geq 1$. We next improve the upper bound in (7.46) for $K \geq 1$. The resulting explicit bound is sharp for $K = 1$.

7.47. Theorem. *For* $n \geq 2$, $K \geq 1$, *and* $0 \leq r \leq 1$

(1) $$\varphi_K(r) \leq \lambda_n^{1-\alpha} r^{\alpha} \, , \quad \alpha = K^{1/(1-n)} \, ,$$

(2) $$\varphi_{1/K}(r) \geq \lambda_n^{1-\beta} r^{\beta} \, , \quad \beta = K^{1/(n-1)} \, .$$

Proof. (1) Let $K \geq 1$, $r \in (0,1)$, and $M(r) = \operatorname{mod} R_{G,n}(1/r)$. Setting $r' = \varphi_K(r)$ we have $M(r') = \alpha M(r)$ and $r' \geq r$. These facts follow from the definitions (7.19) and (7.44).

From the proof of 7.20 it follows that $M(r) + \log r$ is a decreasing function on $(0,1)$, so that

$$M(r) + \log r \geq M(r') + \log r' \, .$$

Let $\lambda_n \geq 4$ be as in (7.21). Since $\log \lambda_n \geq M(r) + \log r$ by (7.21) we obtain

$$0 \leq \log \frac{\lambda_n}{r} - M(r) \leq \log \frac{\lambda_n}{r'} - M(r')$$

and, further, because $0 < \alpha \leq 1$

$$\alpha \log \frac{\lambda_n}{r} - \alpha M(r) \leq \log \frac{\lambda_n}{r'} - M(r') \, .$$

This inequality yields

$$r' \leq \lambda_n^{1-\alpha} r^{\alpha}$$

for $r \in (0,1)$. Because this holds for $r = 0$ and $r = 1$, too, we have completed the proof of (1).

(2) The proof of (2) is similar. □

7.48. Exercise. Applying (7.24) and 7.26(1) show that $\lambda_n \geq 4$. Derive from (7.24) and 7.26 also some inequalities between the constants c_n and ω_{n-1} . [Hint: Note that by (5.57) $\mu(1/\sqrt{2}) = \frac{1}{2}\pi$.] Find also lower bounds for λ_n in terms of ω_{n-1} and c_n .

7.49. Exercise. Show that $\varphi_{K,n}(r) = M_n^{-1}(\alpha M_n(r))$ where $M_n(r) = \operatorname{mod} R_{G,n}(1/r)$ and $\alpha = K^{1/(1-n)}$. For the proof of (7.46) the crude upper bound $\gamma_n(s) \leq \omega_{n-1}(\log s)^{1-n}$ was used. Derive improved versions of (7.46) by using the two upper bounds in 7.26(1). (Note: The resulting inequality will yet be weaker than 7.47.)

It is clear that $r^\alpha \le r^{1/K}$, $\alpha = K^{1/(1-n)}$, for $K \ge 1$ and $0 \le r \le 1$. This fact together with 7.47 and the next lemma, shows that $\varphi_K(r) \le c(K) r^{1/K}$ for $K \ge 1$, where $c(K)$ depends only on K and where $c(K) \to 1$ as $K \to 1$.

7.50. Lemma. *For* $n \ge 2$, $K \ge 1$, *and* $\alpha = K^{1/(1-n)} = 1/\beta$ *the following two inequalities hold:*

$$\lambda_n^{1-\alpha} \le 2^{1-\alpha} K \le 2^{1-1/K} K \,. \tag{1}$$

$$\lambda_n^{1-\beta} \ge 2^{1-\beta} K^{-\beta} \ge 2^{1-K} K^{-K} \,. \tag{2}$$

Proof. (1) It follows from 7.25 that

$$(1-\alpha)\log \lambda_n \le (1-\alpha)(n-1) + (1-\alpha)\log 2 \,.$$

From $1 - e^{-x} \le x$, $x \ge 0$, one can deduce that

$$(1-\alpha)(n-1) = (1 - K^{1/(1-n)})(n-1) \le \log K \,.$$

Because $1 - \alpha \le 1 - 1/K$ we conclude that

$$(1-\alpha)\log \lambda_n \le \log K + (1-\alpha)\log 2 \le \log K + (1-1/K)\log 2$$

and the desired upper bound follows.

(2) The proof of (2) can be derived from (1) as follows

$$\lambda_n^{1-\beta} = \lambda_n^{(\alpha-1)\beta} \ge \left(2^{1-\alpha}K\right)^{-\beta} = 2^{1-\beta}K^{-\beta} \ge 2^{1-K}K^{-K} \,. \quad \square$$

Next we shall prove a "dimension–cancellation" property of the function $\varphi_{K,n}$, $K > 0$, by finding dimension–free minorant and majorant functions.

7.51. Lemma. *For* $K > 0$ *and* $0 < r < 1$ *there exist positive numbers* a_1 *and* a_2 *in* $(0,1)$ *such that* $a_1 \le \varphi_{K,n}(r) \le a_2$ *for all* $n \ge 2$. *In particular,* a_1 *and* a_2 *are independent of* n .

Proof. By 7.26(2) we have

$$A \log \frac{s+1}{s-1} \le \gamma(s) \le A\,\mu\Big(\frac{s-1}{s+1}\Big) \,, \quad A = 2^{n-1} c_n \,, \tag{7.52}$$

for $s > 1$. Because γ is strictly decreasing we obtain from this

$$\gamma^{-1}(t) \le \frac{1 + \mu^{-1}(t/A)}{1 - \mu^{-1}(t/A)}$$

and

$$\gamma^{-1}(t) \geq \operatorname{cth} \frac{t}{2A} .$$

These two inequalities together with (7.52) yield for $r \in (0,1)$

$$b_1 = \operatorname{cth}\Big(\tfrac{1}{2}K\mu\Big(\frac{1-r}{1+r}\Big)\Big) \leq \gamma^{-1}\Big(K\gamma\Big(\frac{1}{r}\Big)\Big) \leq \frac{1+\mu^{-1}(K\log T)}{1-\mu^{-1}(K\log T)} = b_2$$

where $T = (1+r)/(1-r)$. Both bounds are independent of n . In view of (7.44) we may choose $a_1 = 1/b_2$ and $a_2 = 1/b_1$. □

7.53. Exercise. For $n = 2$, $K > 0$, $t > 0$, let $\alpha_K(t) = \tau_2^{-1}(\tau_2(t)/K)$. Show that

$$\alpha_K(t) = \frac{A^2}{1-A^2}\,, \quad A = \varphi_{K,2}\Big(\sqrt{\frac{t}{1+t}}\Big) . \tag{7.54}$$

Let $t_K = 2(4K)^{-K}$. Then for $K \geq 1$ and $t \in (0, t_K]$

$$\varphi_{K,2}(t) \leq \tfrac{1}{2}$$

(see 7.50). Conclude that for $K \geq 1$ and $t \in (0, t_K]$

$$\alpha_K(t) \leq \frac{4}{3} \cdot 16^{1-1/K}\Big(\frac{t}{1+t}\Big)^{1/K} .$$

Next, applying 5.61(2) and 7.47 show that for $K \geq 1$ and $t > 0$

$$\alpha_K(t) \leq 16^{K-1/K}(1+t)^{K-1/K}t^{1/K} .$$

7.55. Exercise. Let G be a uniform domain in $\mathbf{R}^n$ with connected boundary (recall 3.8). Show that if E is a connected subset of G , then $\mathsf{M}(\Delta(E, \partial G)) \geq c\, k_G(E)$ where c is a constant.

7.56. Exercise. (1) Applying the functional identities of μ , one can write the constant b_2 in the proof of 7.51 also in other ways. Show that

$$a_2 = \frac{1}{b_2} = \mu^{-1}\Big(\frac{\pi^2}{2K\log T}\Big) .$$

(2) Find an improved form of 7.50 by replacing the inequality $1 - e^{-x} \leq x$ in the proof of 7.50 by the better inequality $1 - e^{-x} \leq \operatorname{th} x$ (cf. 2.29(1)).

7.57. Remark. The Grötzsch capacity $\gamma_n(s)$ has several interesting properties which are studied in [AVV3]. It is shown there that

$$\gamma_n(1/\operatorname{th}(a+b)) \le \gamma_n(1/\operatorname{th} a) + \gamma_n(1/\operatorname{th} b)$$

holds for all $a, b > 0$. Several inequalities involving the function $\varphi_{K,n}(r)$ can also be found in [AVV1], [AVV2], and [AVV3]. One of these is

$$r^\alpha \le \frac{2r^\alpha}{(1+r')^\alpha + (1-r')^\alpha} \le \varphi_{K,n}(r) \; ; \; r' = \sqrt{1-r^2} \, , \; \alpha = K^{1/(1-n)} \, ,$$

which holds for all $K \ge 1$, $r \in [0,1]$, and $n \ge 2$.

7.58. Exercise. Let $M_n(r)$, $r \in (0,1)$, $n \ge 2$, be as in 7.49. Then $M_2(r) = \mu(r)$. Show that the relationship

$$M_n(r) M_n\big(\sqrt{1-r^2}\,\big) = c \tag{a}$$

for all $r \in (0,1)$ is equivalent to

$$\big[\varphi_{K,n}(r)\big]^2 + \big[\varphi_{1/K,n}\big(\sqrt{1-r^2}\,\big)\big]^2 = 1 \tag{b}$$

for all $r \in (0,1)$ and all $K > 0$. Recall that both (a) and (b) hold for $n = 2$ by (5.57) and 5.61. Next, applying 7.26 show that (a) is false for $n \ge 3$ and, therefore, also (b) is false for $n \ge 3$.

7.59. An open problem. Let E and F be disjoint compact sets in $\mathbf{H}^n$ and let F^* denote the reflection of F in $\partial\mathbf{H}^n$. Consider two curve families $\Gamma = \Delta(E,F)$ and $\Gamma^* = \Delta(E,F^*)$. It seems natural to conjecture that $\mathsf{M}(\Gamma) \ge \mathsf{M}(\Gamma^*)$ ([BBH, p.501, 7.57]). The validity of this conjecture can be verified in certain particular cases, e.g. when E and F are balls. In particular, the conjecture holds true when $n = 2$, as F. W. Gehring and N. Suita have independently shown to the author. Some applications of this fact are given in [LEVU].

7.60. Notes. The method of symmetrization has found many applications in geometry (see [BER, 9.13]) and in various branches of analysis, e.g. in the study of isoperimetric inequalities (see [PS], [BA], [HE2]), and in real analysis. O. Teichmüller [TE] applied these ideas to geometric function theory and proved a special case of Lemma 7.17 above. Other function–theoretic applications are given in [HA2].

In $\mathbf{R}^3$ the conformal capacity was studied by C. Loewner [LO], who applied his result to quasiconformal mappings. Many results of this section are connected with the fundamental results of F. W. Gehring [G1], [G2]. A multidimensional version of Teichmüller's work on symmetrization is contained in [G1] and [S1]. See also [PS]. The literature dealing with p-capacity is vast: the reader is referred to [MK], [FR], [GOR], [MAZ2], and [STR2], [W2], as well as to the bibliographies of these works.

One of the main goals of this section is to find estimates for $\mathsf{M}(\Delta(E,F))$ in terms of geometric quantities such as

$$\frac{\min\{\, d(E),\, d(F)\,\}}{d(E,F)} \;.$$

For 7.34–7.37 see [G1] and [G7]. For 7.41 and 7.42 see [VU10] and [VU13]. The natural setup and motivation for 7.47 is the Schwarz lemma [HP], [WA], [SH], [MRV2], which we shall study in Section 11. For $n=2$ Theorem 7.47 is due to O. Hübner [HÜ] and the same method appears also in [LV2, p. 64] and, in the n-dimensional context, in [AVV1]. For a different proof ($n=2$) see P. P. Belinskiĭ [BEL, p. 15]. Also 7.50 and 7.51 were proved in [AVV1]. For 7.38 see [VU10], [VU13], and [GM1].

From the vast literature dealing with condensers in the plane we mention [B], [KL], [KU], and [T, Ch. III].

8. Conformal invariants

In the preceding sections we have studied some properties of the conformal invariant $\mathsf{M}(\Delta(E,F;G))$. In this section we shall introduce two other conformal invariants, the modulus metric $\mu_G(x,y)$ and its "dual" quantity $\lambda_G(x,y)$, where G is a domain in $\overline{\mathbf{R}}^n$ and $x,y\in G$. The modulus metric μ_G is functionally related to the hyperbolic metric ρ_G if $G=\mathbf{B}^n$, while in the general case μ_G reflects the "capacitary geometry" of G in a delicate fashion. The dual quantity $\lambda_G(x,y)$ is also functionally related to ρ_G if $G=\mathbf{B}^n$. For a wide class of domains in $\mathbf{R}^n$, the so-called QED-domains, we shall find two-sided estimates for $\lambda_G(x,y)$ in terms of

$$r_G(x,y)=\frac{|x-y|}{\min\{\, d(x,\partial G),\, d(y,\partial G)\,\}} \;.$$

8.1. The conformal invariants λ_G and μ_G. If G is a proper subdomain of $\overline{\mathbf{R}}^n$, then for $x,y \in G$ with $x \neq y$ we define

$$\lambda_G(x,y) = \inf_{C_x, C_y} \mathsf{M}(\Delta(C_x, C_y; G)) \tag{8.2}$$

where $C_z = \gamma_z[0,1)$ and $\gamma_z\colon [0,1) \to G$ is a curve such that $z \in |\gamma_z|$ and $\gamma_z(t) \to \partial G$ when $t \to 1$, $z = x, y$. It follows from 5.17 that λ_G is invariant under conformal mappings of G. That is, $\lambda_{fG}(f(x), f(y)) = \lambda_G(x,y)$, if $f\colon G \to fG$ is conformal and $x, y \in G$ are distinct.

8.3. Remark. If $\operatorname{card}(\overline{\mathbf{R}}^n \setminus G) = 1$, then $\lambda_G(x,y) \equiv \infty$ by 5.33. Therefore λ_G is of interest only in case $\operatorname{card}(\overline{\mathbf{R}}^n \setminus G) \geq 2$. For $\operatorname{card}(\overline{\mathbf{R}}^n \setminus G) \geq 2$ and $x, y \in G$, $x \neq y$, there are continua C_x and C_y as in (8.2) with $\overline{C}_x \cap \overline{C}_y = \emptyset$ and thus $\mathsf{M}(\Delta(C_x, C_y; G)) < \infty$ by 5.23. Thus, if $\operatorname{card}(\overline{\mathbf{R}}^n \setminus G) \geq 2$, we may assume that the infimum in (8.2) is taken over continua C_x and C_y with $\overline{C}_x \cap \overline{C}_y = \emptyset$.

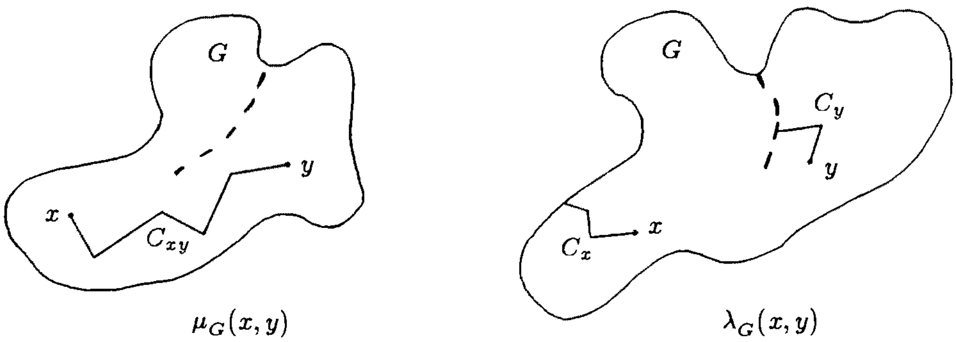


Diagram 8.1.

For a proper subdomain G of $\overline{\mathbf{R}}^n$ and for all $x, y \in G$ define

$$\mu_G(x,y) = \inf_{C_{xy}} \mathsf{M}(\Delta(C_{xy}, \partial G; G)) \tag{8.4}$$

where the infimum is taken over all continua C_{xy} such that $C_{xy} = \gamma[0,1]$ and γ is a curve with $\gamma(0) = x$ and $\gamma(1) = y$. It is clear that μ_G is also a conformal invariant in the same sense as λ_G. It is left as an easy exercise for the reader to verify that μ_G is a metric if $\operatorname{cap} \partial G > 0$. [Hint: Apply 5.9 and 6.1.] If $\operatorname{cap} \partial G > 0$, we call μ_G the *modulus metric* or *conformal metric* of G.

8.5. Remark. Let D be a subdomain of G. It follows from 5.9 and (5.10) that $\mu_G(a,b) \le \mu_D(a,b)$ for all $a,b \in D$ and $\lambda_G(a,b) \ge \lambda_D(a,b)$ for all distinct $a,b \in D$. In what follows we are interested only in the non-trivial case $\mathrm{card}(\overline{\mathbf{R}}^n \setminus G) \ge 2$. Moreover, by performing an auxiliary Möbius transformation, we may and shall assume that $\infty \in \overline{\mathbf{R}}^n \setminus G$ throughout this section. Hence G will have at least one finite boundary point.

In a general domain G, the values of $\lambda_G(x,y)$ and $\mu_G(x,y)$ cannot be expressed in terms of well-known simple functions. For $G = \mathbf{B}^n$ they can be given in terms of $\rho(x,y)$ and the capacity of the Teichmüller condenser.

8.6. Theorem. *The following identities hold for all distinct* $x,y \in \mathbf{B}^n$ *:*

(1) $$\mu_{\mathbf{B}^n}(x,y) = 2^{n-1}\tau\Big(\frac{1}{\mathrm{sh}^2\,\frac{1}{2}\rho(x,y)}\Big) = \gamma\Big(\frac{1}{\mathrm{th}\,\frac{1}{2}\rho(x,y)}\Big)\,,$$

(2) $$\lambda_{\mathbf{B}^n}(x,y) = \tfrac{1}{2}\tau\big(\mathrm{sh}^2\,\tfrac{1}{2}\rho(x,y)\big)\,.$$

Proof. (1) The proof of part (1) follows directly from 7.27, (7.32), and 5.53.

(2) Because the assertion is $\mathcal{GM}(\mathbf{B}^n)$-invariant, we may assume that $x = re_1 = -y$ and $r = \mathrm{th}\big(\frac{1}{4}\rho(x,y)\big)$ (see (2.25)). By a symmetry property 5.20 of the modulus and by 5.54(1) we obtain

$$\begin{aligned}\lambda_{\mathbf{B}^n}(x,y) &\le \mathsf{M}\big(\Delta([-e_1,-re_1],[re_1,e_1];\mathbf{B}^n)\big)\\ &= \tfrac{1}{2}\,\mathsf{M}\big(\Delta([-\tfrac{1}{r}e_1,-re_1],[re_1,\tfrac{1}{r}e_1];\mathbf{R}^n)\big)\\ &= \tfrac{1}{2}\,\tau\Big(\frac{4r^2}{(1-r^2)^2}\Big) = \tfrac{1}{2}\,\tau\,\big(\mathrm{sh}^2\,\tfrac{1}{2}\rho(x,y)\big)\,.\end{aligned}$$

Hence it will suffice to prove the inequality "$\ge$".

Let C_x, C_y be as in (8.2) and $0 < \epsilon < \frac{1}{2}(1-|x|)$. Choose compact connected subsets E, F of C_x, C_y with $x \in E$, $y \in F$ and $d(E,S^{n-1}) = d(F,S^{n-1}) = \epsilon$. Let $E^s = E \cup hE$, $F^s = F \cup hF$ where $h(x) = x/|x|^2$. By 8.3 we may assume that $\overline{C}_x \cap \overline{C}_y = \emptyset$ and hence at most one of the sets E and F can contain 0. We may assume $0 \notin F$, hence F^s is compact. Let $\mathrm{Sym}(F^s)$ denote the set obtained from F^s by spherical symmetrization in the positive x_1-axis and let $\mathrm{Sym}(E^s)$ be the set obtained from E^s by spherical symmetrization in the negative x_1-axis. By 7.17, 7.8, and 5.9

$$\begin{aligned}&\mathrm{cap}(\mathbf{R}^n \setminus E^s, F^s) \ge \mathrm{cap}(\mathbf{R}^n \setminus \mathrm{Sym}(E^s), \mathrm{Sym}(F^s))\\ &\ge \mathsf{M}\big(\Delta([-\tfrac{1}{r}e_1,-re_1],[re_1,\tfrac{1}{r}e_1])\big) - 2\,\mathsf{M}\big(\Delta(Y_1,Y_2)\big)\end{aligned}$$

where $Y_1 = [-\frac{1}{r}e_1,\ -re_1]$ and $Y_2 = [(1-\epsilon)e_1,\ (1-\epsilon)^{-1}e_1]$. This inequality together with 5.54(1) yields

$$\operatorname{cap}(\mathbf{R}^n \setminus E^s\,,\, F^s) \ge \tau\big(\operatorname{sh}^2 \tfrac{1}{2}\rho(x,y)\big) - \delta(\epsilon)$$

where $\delta(\epsilon) \to 0$ as $\epsilon \to 0$. Letting $\epsilon \to 0$ and applying 5.20 yields $\mathsf{M}(\Delta(C_x, C_y; \mathbf{B}^n)) \ge \frac{1}{2}\tau(\operatorname{sh}^2 \frac{1}{2}\rho(x,y))$. Since C_x and C_y were arbitrary sets with the stated properties, the desired inequality $\lambda_{\mathbf{B}^n}(x,y) \ge \frac{1}{2}\tau(\operatorname{sh}^2 \frac{1}{2}\rho(x,y))$ follows. □

8.7. Remark. From 7.26(3) we obtain the following inequality for $x, y \in \mathbf{B}^n$ (exercise)

$$\begin{aligned}\tfrac{1}{2}\tau\big(\operatorname{sh}^2 \tfrac{1}{2}\rho(x,y)\big) &\ge -c_n \log \operatorname{th} \tfrac{1}{4}\rho(x,y)\\ &= 2c_n \operatorname{arth}\left(e^{-\frac{1}{2}\rho(x,y)}\right) \ge 2c_n e^{-\frac{1}{2}\rho(x,y)}\,.\end{aligned}$$

Here the identities $2\operatorname{ch}^2 A = 1 + \operatorname{ch} 2A$ and $\operatorname{sh} 2A = 2 \operatorname{ch} A \operatorname{sh} A$ were applied (see also 2.29(3)). Recall that

$$\operatorname{sh}^2 \tfrac{1}{2}\rho(x,y) = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}$$

by (2.19). Similarly, by 7.26(3) we obtain also

$$\begin{aligned}\tfrac{1}{2}\tau\big(\operatorname{sh}^2 \tfrac{1}{2}\rho(x,y)\big) &\le \tfrac{1}{2}c_n\, \mu\big(\operatorname{th}^2(\tfrac{1}{4}\rho(x,y))\big) < \tfrac{1}{2}c_n \log \frac{4}{\operatorname{th}^2 \frac{1}{4}\rho(x,y)}\\ &= c_n \log \frac{2}{\operatorname{th} \frac{1}{4}\rho(x,y)}\,.\end{aligned}$$

8.8. Lemma. *Let G be a proper subdomain of $\mathbf{R}^n$, $x \in G$, $d(x) = d(x, \partial G)$, $B_x = B^n(x, d(x))$, let $y \in B_x$ with $y \ne x$, and let $r = |x-y|/d(x)$. Then the following two inequalities hold:*

(1) $$\lambda_G(x,y) \ge \lambda_{B_x}(x,y) = \tfrac{1}{2}\,\tau\Big(\frac{r^2}{1-r^2}\Big) > c_n \log \frac{1}{r}\ ,$$

(2) $$\mu_G(x,y) \le \mu_{B_x}(x,y) = \gamma\Big(\frac{1}{r}\Big) \le \omega_{n-1}\Big(\log \frac{1}{r}\Big)^{1-n}\,.$$

Proof. (1) By 8.5, 8.6(2), and 8.7 we obtain

$$\begin{aligned}\lambda_G(x,y) \ge \lambda_{B_x}(x,y) &= \tfrac{1}{2}\,\tau\Big(\frac{r^2}{1-r^2}\Big) \ge -c_n \log \operatorname{th} \tfrac{1}{4}(2 \operatorname{arth} r)\\ &= c_n \log \frac{1+\sqrt{1-r^2}}{r} > c_n \log \frac{1}{r}\end{aligned}$$

(2) The desired inequalities follow from 8.5 and (7.24). □

8.9. The function $p(x)$. For $x \in \mathbf{R}^n \setminus \{0, e_1\}$, $n \geq 2$, define

$$p(x) = \inf_{E,F} \mathsf{M}(\Delta(E,F)) \tag{8.10}$$

where the infimum is taken over all pairs of continua E and F in $\overline{\mathbf{R}}^n$ with the properties $0,\, e_1 \in E$ and $x, \infty \in F$.

8.11. Lemma. *The inequality*

$$p(x) \geq \max\{\, \tau(|x|)\, ,\, \tau(|x - e_1|)\}$$

holds for all $x \in \mathbf{R}^n \setminus \{0, e_1\}$. *Equality holds if* $x = se_1$ *and* $s < 0$ *or* $s > 1$.

Proof. The proof follows directly from 7.17. □

The main result of this section is the following theorem.

8.12. Theorem. *For* $|x - e_1| \leq |x|$, $x \in \mathbf{R}^n \setminus \{0, e_1\}$

(1) $p(x) \leq 2\,\tau(|x - e_1|)$ *when* $|x + e_1| \geq 2$,

(2) $p(x) \leq 4\,\tau(|x - e_1|)$ *when* $|x| \geq 1$,

(3) $p(x) \leq 2^{n+1}\tau(|x - e_1|)$.

The proof of Theorem 8.12 will be divided into several parts. Due to symmetry properties of the above definition (8.10) (that is axial symmetry in the x_1-axis and symmetry in the $(n-1)$-dimensional plane $x_1 = \frac{1}{2}$) it is clear that the values of $p(x)$ are determined by its values in the set

$$D_1 = \{\, (x_1, 0, \ldots, 0, x_n) : x_1 \geq \tfrac{1}{2},\, x_n \geq 0\} \setminus \{e_1\}\,. \tag{8.13}$$

All the upper bounds (1)–(3) in Theorem 8.12 are based on Lemma 5.27 and on the functional inequalities of $\tau(s)$ in Lemma 5.63.

8.14. Lemma. *If* $x \in \mathbf{R}^n \setminus B^n(-2e_1, 3)$, *then*

$$4(|x| - 1) \geq \min\{\, |x - e_1|\, ,\, |x - e_1|^2\,\}\,.$$

Proof. Write $x = x + 2e_1 - 2e_1$ and $x - e_1 = x + 2e_1 - 3e_1$. Then by the law of cosines

$$|x|^2 = |x + 2e_1|^2 + 4 - 4(x + 2e_1) \cdot e_1\,,$$

$$|x - e_1|^2 = |x + 2e_1|^2 + 9 - 6(x + 2e_1) \cdot e_1\,.$$

From this we obtain

$$3|x|^2 - 2|x-e_1|^2 = |x+2e_1|^2 - 6 \geq 9 - 6 = 3 \, .$$

Hence $|x| > (1+\frac{2}{3}|x-e_1|^2)^{1/2}$, so that

$$|x| - 1 \geq \frac{\frac{2}{3}|x-e_1|^2}{1+\sqrt{1+\frac{2}{3}|x-e_1|^2}} \, .$$

Case A. $|x-e_1| \leq 1$. Then

$$|x| - 1 \geq \frac{\frac{2}{3}|x-e_1|^2}{1+\sqrt{1+\frac{2}{3}}} \geq \frac{1}{4}|x-e_1|^2 \, .$$

Case B. $|x-e_1| > 1$. Then

$$|x| - 1 \geq \frac{\frac{2}{3}|x-e_1||x-e_1|}{1+\sqrt{1+\frac{2}{3}|x-e_1|^2}} \geq \frac{\frac{2}{3}|x-e_1|}{1+\sqrt{1+\frac{2}{3}}} > \frac{1}{4}|x-e_1| \, ,$$

since $t \mapsto t/(1+\sqrt{1+\frac{2}{3}t^2})$ is increasing on $(0,\infty)$.

The proof follows from the above inequalities. ❐

8.15. Lemma. *Let* $E = [0, e_1]$ *and* $F = [x, \infty]$ *for* $x \in \mathbf{R}^n \setminus \mathbf{B}^n$. *Then*

$$\text{(1)} \qquad p(x) \leq \mathsf{M}(\Delta(E,F)) \leq \tau(|x|-1) \, .$$

If $x \in \mathbf{R}^n \setminus B^n(-2e_1, 3)$, *then*

$$\text{(2)} \qquad p(x) \leq \mathsf{M}(\Delta(E,F)) \leq 2\,\tau(|x-e_1|) \, .$$

Proof. The inequality (1) follows from 5.27. It follows from 5.63(2) that $\tau(u) \leq 2\tau(2u+2\sqrt{u}) < 2\tau(2\sqrt{u})$ and hence $\tau(\frac{1}{4}s^2) < 2\tau(s)$. From 5.63(2) it also follows that $\tau(\frac{1}{4}s) < 2\tau(s)$. In conclusion, for $s > 0$ the following inequality holds

$$\tau\big(\tfrac{1}{4}\min\{s, s^2\}\big) < 2\tau(s) \, .$$

The proof of the inequality (2) follows from (1), the above inequality, and from Lemma 8.14. ❐

8.16. Exercise. For $0 \le \alpha < \frac{1}{2}\pi$ let $x_t = e_1 + t\big((\cos\alpha)e_1 + (\sin\alpha)e_n\big)$, $t > 0$. For fixed α and arbitrary $t > 0$ show that

$$p(x_t) \le \tau(t\cos\alpha) .$$

8.17. Proof of Theorem 8.12(1). Let $Y = \{ x \in S^{n-1}(-e_1, 2) : x_1 = \frac{1}{2} \}$. Note that $d(e_1, Y) = \sqrt{2}$. It suffices to prove the result for $x \in D_1 \setminus B^n(-e_1, 2)$.

Case A. $|x - e_1| \le \sqrt{2}$.

Choose $\overline{x} \in S^{n-1}(-e_1, 2) \cap D_1$ with $|\overline{x} - e_1| = |x - e_1|$.

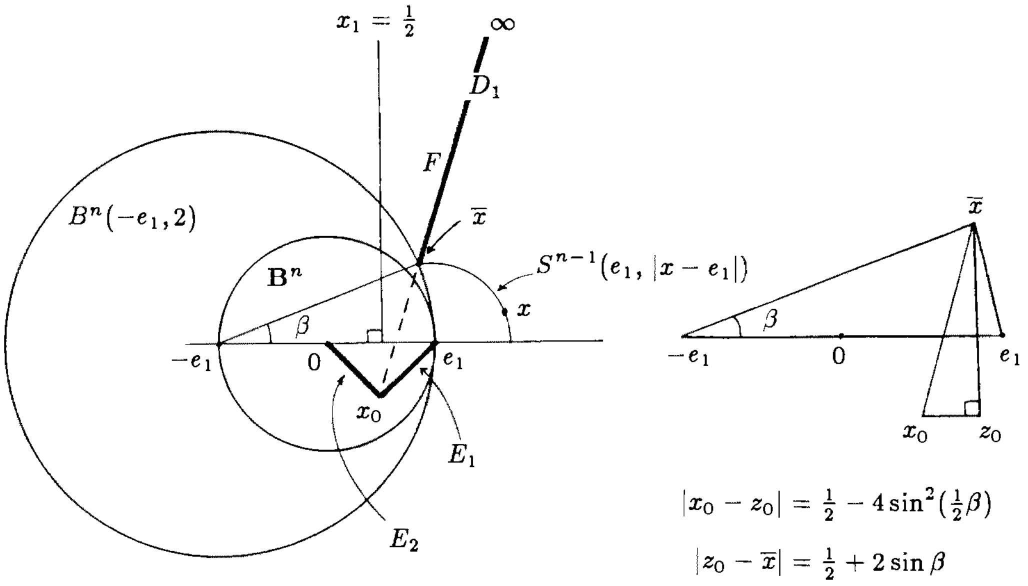


Diagram 8.2.

Then $|\overline{x} - e_1| = 4\sin\frac{1}{2}\beta$ where β is the acute angle between the segments $[-e_1, e_1]$ and $[-e_1, \overline{x}]$. Let $x_0 = \frac{1}{2}(e_1 - e_n)$. Then

$$\begin{aligned} |\overline{x} - x_0|^2 &= (2\sin\beta + \tfrac{1}{2})^2 + (\tfrac{1}{2} - 4\sin^2\tfrac{1}{2}\beta)^2 \\ &= \tfrac{1}{2} + 12\sin^2\tfrac{1}{2}\beta + 2\sin\beta = \tfrac{1}{2}(1 + A) \end{aligned}$$

where $A = 24\sin^2\frac{1}{2}\beta + 4\sin\beta$. Hence

$$\frac{|\overline{x} - x_0|}{|x_0 - e_1|} - 1 = \frac{A}{1 + \sqrt{1 + A}} .$$

It is left as an exercise for the reader to show that

$$\frac{A}{1+\sqrt{1+A}} \geq |\overline{x}-e_1| = 4\sin\tfrac{1}{2}\beta$$

holds for all x in $D_1 \setminus B^n(-e_1,2)$. Let $E_1 = [x_0,e_1]$, $E_2 = [0,x_0]$, and $F = \{\, x_0 + t(\overline{x}-x_0) : t \geq 1 \,\}$. By 5.27 and the last two inequalities we obtain

$$\mathsf{M}\big(\Delta(E_j,F)\big) \leq \tau\Big(\frac{|\overline{x}-x_0|}{|x_0-e_1|} - 1\Big) \leq \tau(|\overline{x}-e_1|)\ .$$

Because $|\overline{x}-e_1| = |x-e_1|$, we obtain by 5.27 and 5.9

$$p(x) \leq \mathsf{M}\big(\Delta(E_1 \cup E_2, F)\big) \leq 2\,\tau(|x-e_1|)$$

as desired. Note that the condition $|x-e_1| \leq \sqrt{2}$ was used only for the construction of $\overline{x}$.

Case B. $|x-e_1| \geq \sqrt{2}$.

It is easy to see that in this case for $x \in D_1$

$$\frac{|x|-1}{|x-e_1|} \geq \frac{\sqrt{2}-1}{\sqrt{2}} \geq \frac{1}{4}$$

and hence by 8.14(1)

$$p(x) \leq \tau(|x|-1) \leq \tau\big(\tfrac{1}{4}|x-e_1|\big)\ .$$

Because $\tau(s) \leq 4\tau(4s)$ by 5.63(2) we obtain the desired inequality also in Case B. □

8.18. Exercise. Show that $A/(1+\sqrt{1+A}) \geq 4\sin\frac{1}{2}\beta$ in the above proof.

8.19. Theorem. *For* $x \in D_1 \setminus B^n\big(-\frac{1}{2}(e_1 + 3e_n/\tan\alpha), 3/(2\tan\alpha)\big)$ *and* $0 < \alpha < \frac{1}{2}\pi$, *the following inequality holds*

$$p(x) \leq 4\,\tau\big(2(\sin\alpha)|x-e_1|\big)\ .$$

Proof. Let $x_0 = \frac{1}{2}(e_1 - e_n/\tan\alpha)$. Let $E_1 = [x_0,e_1]$, $E_2 = [0,x_0]$, $F = \{\, x_0 + t(x-x_0) : t \geq 1 \,\}$, $\Gamma_j = \Delta(E_j,F)$, $j = 1,2$. It follows from 8.15(1) that

$$\mathsf{M}(\Gamma_j) \leq \tau\Big(\frac{|x-x_0|}{|x_0-e_1|} - 1\Big)$$

for $j = 1, 2$. Because of the choice of x, 8.15 yields

$$\tau\Big(\frac{|x-x_0|}{|x_0-e_1|}-1\Big) \le 2\,\tau\Big(\frac{|x-e_1|}{|x_0-e_1|}\Big) = 2\tau\big((2\sin\alpha)|x-e_1|\big)\,.$$

These inequalities together with 5.9 yield the desired bound

$$p(x) \le \mathsf{M}\big(\Delta(E_1\cup E_2, F)\big) \le 4\,\tau\big((2\sin\alpha)|x-e_1|\big)\,. \quad \square$$

8.20. Proof of Theorem 8.12(2). We may assume that $x \in D_1 \setminus \mathbf{B}^n$. Let $\alpha = \frac{1}{4}\pi$ and $x_0 = \frac{1}{2}(e_1 - e_n/\tan\alpha)$. As in the proof of Theorem 8.19 we get

$$p(x) \le 4\,\tau\big(\sqrt{2}\,|x-e_1|\big) < 4\,\tau(|x-e_1|)$$

as desired. $\square$

8.21. Proof of Theorem 8.12(3). We may assume that $x \in D_1$. If $x_n \ge \frac{1}{3}\sqrt{3}(1-x_1)$, then Lemma 8.19 with $\alpha = \pi/6$ yields

$$p(x) \le 4\,\tau(|x-e_1|)\,. \tag{8.22}$$

If $x_n < \frac{1}{3}\sqrt{3}(1-x_1)$, choose $\overline{x} \in D_1$ with $\overline{x}_1 = x_1$ and $\overline{x}_n = \frac{1}{3}\sqrt{3}(1-\overline{x}_1)$. Let $x_0 = \frac{1}{2}(e_1 - \sqrt{3}\,e_n)$,

$$E = [0, x_0] \cup [x_0, e_1]\,,$$

and

$$F = [x, \overline{x}] \cup \{\, x_0 + t(\overline{x} - x_0) : t \ge 1 \,\}\,.$$

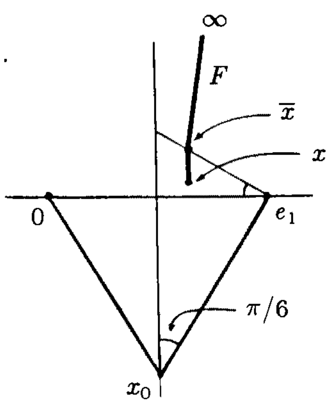


Diagram 8.3.

By (8.22) and 5.53

$$\begin{aligned} p(x) &\le \mathsf{M}\big(\Delta(E,F)\big) < 4\,\tau(|\overline{x}-e_1|) + \mathsf{M}\big(\Delta([x,\overline{x}],E)\big) \\ &\le 4\,\tau(|x-e_1|) + \mathsf{M}(\Gamma) \le 4\,\tau(|x-e_1|) + \gamma(2) \\ &= 4\,\tau(|x-e_1|) + 2^{n-1}\tau(3) \le 2\max\{4, 2^{n-1}\}\,\tau(|x-e_1|) \\ &\le 2^{n+1}\tau(|x-e_1|) \end{aligned}$$

where $\Gamma = \Delta\big([x,\overline{x}]\,, S^{n-1}(\overline{x}, |\overline{x}-e_1|)\big)$. $\square$

For $x \in \mathbf{R}^n \setminus \{0\}$ we denote by r_x a similarity map with $r_x(0) = 0$ and $r_x(x) = e_1$. Then it is easy to see that $|r_x(y) - e_1| = |x-y|/|x|$. It follows immediately from the definitions (8.2) and (8.10) that

$$\lambda_{\mathbf{R}^n\setminus\{0\}}(x,y) = \min\{\, p(r_x(y))\,,\, p(r_y(x))\,\}\,. \tag{8.23}$$

Next we deduce the following two–sided inequality for $\lambda_{\mathbf{R}^n\setminus\{0\}}(x,y)$.

8.24. Theorem. *For distinct* $x, y \in \mathbf{R}^n \setminus \{0\}$ *the following inequality holds*

$$1 \le \lambda_{\mathbf{R}^n\setminus\{0\}}(x,y)/\tau\big(|x-y|/\min\{|x|,|y|\}\big) \le 4\,.$$

Proof. We may assume that $|x| \le |y|$. Denote $G = \mathbf{R}^n \setminus \{0\}$.

We prove first the lower bound. By (8.23) and 8.11 we get

$$\begin{aligned}\lambda_G(x,y) &\ge \min\{\,\tau(|r_x(y) - e_1|)\,,\,\tau(|r_y(x) - e_1|)\,\}\\ &= \tau\big(\max\{\,|x-y|/|x|\,,\,|x-y|/|y|\,\}\big) = \tau(|x-y|/|x|)\,,\end{aligned}$$

which is the desired lower bound.

For the proof of the upper bound let V be the $(n-1)$–dimensional plane orthogonal to $[0,x]$ at $\frac{1}{2}x$ and let H_0, H_x be the components of $\mathbf{R}^n \setminus V$, $x \in H_x$. Consider two cases.

Case A. $y \in \overline{H}_x$. Because $|y| \ge |x|$ it follows from 8.12(2) that

$$\lambda_G(x,y) \le p(r_x(y)) \le 4\,\tau(|x-y|/|x|)\,.$$

Case B. $y \in H_0$. Let $E_1 = [0, \frac{1}{2}x]$, $E_2 = [\frac{1}{2}x, x]$, and $F = \{\,\frac{1}{2}x + t(y - \frac{1}{2}x) : t \ge 1\,\}$, $\Gamma_j = \Delta(E_j, F)$, $j = 1,2$. Then by 5.27

$$\mathsf{M}(\Gamma_j) \le \tau\Big(\frac{2|y - \frac{1}{2}x|}{|x|} - 1\Big)$$

for $j = 1,2$. Because $|y - \frac{1}{2}x| \ge \frac{1}{2}\sqrt{3}|y|$ and $|y| \ge |x|$ we obtain

$$\begin{aligned}\lambda_G(x,y) &\le \mathsf{M}(\Gamma_1) + \mathsf{M}(\Gamma_2) \le 2\,\tau\Big(\frac{\sqrt{3}\,|y|}{|x|} - 1\Big)\\ &\le 2\,\tau\Big((\sqrt{3}-1)\frac{|y|}{|x|}\Big) \le 2\,\tau\Big(\tfrac{1}{2}(\sqrt{3}-1)\frac{|x-y|}{|x|}\Big)\,.\end{aligned}$$

From 5.63 it follows that $\tau(s) \le 2\,\tau(4s)$ and hence the above inequalities imply

$$\lambda_G(x,y) \le 4\,\tau\Big(2(\sqrt{3}-1)\frac{|x-y|}{|x|}\Big) \le 4\,\tau\Big(\frac{|x-y|}{|x|}\Big)$$

as desired. ☐

The above results provide us with some efficient estimates for λ_G, which we now give.

8.25. Corollary. *Let G be a proper subdomain of $\mathbf{R}^n$, x and y distinct points in G and $m(x,y) = \min\{d(x), d(y)\}$. Then*

$$\lambda_G(x,y) \le \inf_{z\in\partial G} \lambda_{\mathbf{R}^n\setminus\{z\}}(x,y) \le 4\,\tau\big(|x-y|/m(x,y)\big) .$$

Proof. The first inequality follows from 8.5. For the second one fix $z_0 \in \partial G$ with $m(x,y) = d(\{x,y\},\{z_0\})$. Applying 8.24 to $\mathbf{R}^n \setminus \{z_0\}$ yields the desired result. □

We next show that 8.25 fails to be sharp for a Jordan domain G in $\mathbf{R}^n$. For $t \in (0, \frac{1}{5})$ consider the family $G_t = B^n(-e_1,1) \cup B^n(e_1,1) \cup B^n(t)$ of Jordan domains. Then by 8.25

$$\lambda_{G_t}(-e_1,e_1) \le 4\,\tau(2)$$

for all $t \in (0, \frac{1}{5})$. But this is far from sharp because in fact

$$\begin{aligned}\lambda_{G_t}(-e_1,e_1) &\le \mathsf{M}\big(\Delta([-2e_1,-e_1],[e_1,2e_1];G_t)\big)\\ &\le \omega_{n-1}\Big(\log\frac{1}{t}\Big)^{1-n} \longrightarrow 0\end{aligned}$$

as $t \to 0$. However, for a wide class of domains, which we shall now consider, the upper bound in 8.25 is essentially best possible.

8.26. QED **domains.** A closed set E in $\overline{\mathbf{R}}^n$ is called a *c–quasiextremal distance set* or *c-QED exceptional set* or *c-QED set*, $c \in (0,1]$, if for each pair of disjoint continua $F_1, F_2 \subset \overline{\mathbf{R}}^n \setminus E$

$$\mathsf{M}\big(\Delta(F_1,F_2;\overline{\mathbf{R}}^n \setminus E)\big) \ge c\,\mathsf{M}\big(\Delta(F_1,F_2)\big) . \tag{8.27}$$

If G is a domain in $\overline{\mathbf{R}}^n$ such that $\overline{\mathbf{R}}^n \setminus G$ is a c-QED set, then we call G a c-QED domain. If $c = 1$ then the set E is called a *null set for extremal distances* or a NED set.

8.28. Examples. (1) The unit ball $\mathbf{B}^n$ is a $\frac{1}{2}$–QED set by [GM1] or by the above Lemma 5.22.

(2) If E is a compact set of capacity zero, then E is a 1–QED set. For instance all isolated sets are 1–QED sets. The class of all 1–QED sets contains all closed sets in $\mathbf{R}^n$ of vanishing $(n-1)$–dimensional Hausdorff measure (see [V3], [GM1]).

(3) $\mathbf{B}^2 \setminus [0,e_1)$ is not a c-QED set for any $c > 0$.

8.29. Theorem. *Let G be a c-QED domain in $\mathbf{R}^n$. Then*

$$\lambda_G(x,y) \geq c\,\tau(s^2+2s) \geq 2^{1-n}c\,\tau(s)$$

where $s = |x-y|/\min\{d(x),\, d(y)\}$.

Proof. Let C_x and C_y be connected sets as in (8.2) with $x \in C_x$ and $y \in C_y$. Let $\Gamma_1 = \Delta(C_x, C_y; G)$ and $\Gamma_2 = \Delta(C_x, C_y)$. We may assume that $d(x,\partial G) \leq d(y,\partial G)$. Fix $u \in \overline{C}_x$ and $v \in \overline{C}_y$ with $|x-u| = d(x,\partial G)$ and $|y-v| = d(y,\partial G) \geq d(x,\partial G)$. Because $|u-v| \leq |u-x| + |x-y| + |y-v|$ we obtain by 7.26 and 5.63(1)

$$\begin{aligned}
\mathsf{M}(\Gamma_1) &\geq c\,\mathsf{M}(\Gamma_2) \geq c\,\tau\Big(\frac{|x-y|\,|u-v|}{|x-u|\,|y-v|}\Big) \\
&\geq c\,\tau\left(|x-y|\Big(\frac{1}{|y-v|} + \frac{|x-y|}{|x-u|\,|y-v|} + \frac{1}{|x-u|}\Big)\right) \\
&\geq c\,\tau(s^2+2s) \geq c\,\tau(4s^2+4s) \geq 2^{1-n}c\,\tau(s)
\end{aligned}$$

as desired. □

It should be noted that the lower bound of 8.29 is very close to that of 8.24; in fact it differs only by a multiplicative constant.

In the next few theorems we shall give some estimates for the conformal metric μ_G.

8.30. Lemma. *Let G be a proper subdomain of $\mathbf{R}^n$, $s \in (0,1)$, $x,y \in G$. If $k_G(x,y) \leq 2\log(1+s)$, then*

(1) $$\mu_G(x,y) \leq \gamma\Big(\frac{1}{\operatorname{th}(k_G(x,y)/(1-s))}\Big)\,.$$

Moreover, there exist positive numbers b_1 and b_2 depending only on n such that

(2) $$\mu_G(x,y) \leq b_1 k_G(x,y) + b_2$$

for all $x,y \in G$.

Proof. (1) Choose a quasihyperbolic geodesic segment $J_G[x,y]$ connecting x to y and let $z \in J_G[x,y]$ with $k_G(x,y) = 2k_G(x,z) = 2k_G(y,z)$. Then by (3.4)

$$j_G(x,z) = \log\Big(1 + \frac{|x-z|}{\min\{d(x),\, d(z)\}}\Big) \leq k_G(x,z) \leq \log(1+s)$$

and hence $x \in \overline{B}^n(z, s\,d(z))$. Let $B_z = B^n(z, d(z))$. By 3.7(1), (3.6), and (3.4) we obtain

$$\begin{aligned} k_{B_z}(x,z) &\le \log\Big(1+\frac{|x-z|}{d(z)-|x-z|}\Big) \le \log\Big(1+\frac{|x-z|}{(1-s)d(z)}\Big) \\ &\le \frac{1}{1-s}\log\Big(1+\frac{|x-z|}{d(z)}\Big) \le \frac{1}{1-s}\, j_G(x,z) \le \frac{1}{1-s}\, k_G(x,z)\,. \end{aligned}$$

Because of the symmetric choice of the point z, we get a similar upper bound also for $k_{B_z}(z,y)$. Hence

$$\begin{aligned} k_{B_z}(x,y) &\le k_{B_z}(x,z) + k_{B_z}(z,y) \\ &\le \frac{1}{1-s}\big(k_G(x,z)+k_G(z,y)\big) = \frac{1}{1-s}\, k_G(x,y)\,. \end{aligned}$$

Denote by ρ_{B_z} the hyperbolic metric of B_z (see 4.25). Now by 3.3 and the above results we get

$$\rho_{B_z}(x,y) \le 2\,k_{B_z}(x,y) \le \frac{2}{1-s}\, k_G(x,y)$$

and hence by 8.5, 8.6(1), and 5.53

$$\begin{aligned} \mu_G(x,y) \le \mu_{B_z}(x,y) &= 2^{n-1}\tau\Big(\frac{1}{\mathrm{sh}^2\,\frac{1}{2}\rho(x,y)}\Big) \\ &= \gamma\Big(\frac{1}{\mathrm{th}\,\frac{1}{2}\rho(x,y)}\Big) \le \gamma\Big(\frac{1}{\mathrm{th}(k_G(x,y)/(1-s))}\Big) \end{aligned}$$

where $\rho = \rho_{B_z}$.

(2) Choose points $x_1, \ldots, x_{p+1} \in J_G[x,y]$ with $x_1 = x$, $x_{p+1} = y$ and $k_G(x_j, x_{j+1}) = 2\log(1+s)$ for $j = 1, \ldots, p-1$ and $k_G(x_p, x_{p+1}) < 2\log(1+s)$ and $p \le 1 + k_G(x,y)/(2\log(1+s))$ (cf. the proof of Lemma 4.9(1)). Then by part (1)

$$\mu_G(x,y) \le \sum_{j=1}^{p} \mu_G(x_j, x_{j+1}) \le p\, b_2$$

where $b_2 = \gamma(1/\mathrm{th}[(2\log(1+s))/(1-s)])$. The desired result with $b_1 = b_2/(2\log(1+s))$ follows. ☐

It should be observed that Lemma 8.30(2) is a generalization of the upper bound in (7.31) to the case of an arbitrary domain. The lower bound in (7.31) will next be generalized to the case of domains with connected boundary.

8.31. Lemma. *Let G be a domain in $\mathbf{R}^n$ such that ∂G is connected. Then for all $a, b \in G$, $a \neq b$,*

$$\mu_G(a,b) \geq \tau(4m^2 + 4m) \geq c_n \, j_G(a,b) \tag{1}$$

where c_n is the constant in 5.28 and $m = \min\{d(a),\, d(b)\}/|a-b|$. If, in addition, G is uniform, then

$$\mu_G(a,b) \geq B\, k_G(a,b) \tag{2}$$

for all $a, b \in G$.

Proof. Statement (1) follows from 7.38 and (8.4), while (2) follows from (1) and 3.8. □

The above results in 8.29 and 8.31 are invariant under similarities but not under $\mathcal{GM}(\overline{\mathbf{R}}^n)$. This is an aesthetic flaw; since λ_G and μ_G are conformal invariants one would naturally expect conformally invariant results. Next we proceed to give bounds for λ_G and μ_G in terms of conformally invariant majorant/minorant functions.

For distinct a, b, c, d in $\overline{\mathbf{R}}^n$ let

$$m(a,b,c,d) = \max\{\, |a,b,d,c\,|,\ |a,c,d,b\,|\,\}. \tag{8.32}$$

If $G \subset \overline{\mathbf{R}}^n$ is a domain with $\mathrm{card}(\overline{\mathbf{R}}^n \setminus G) \geq 2$ then let

$$m_G(b,c) = \sup\{\, m(a,b,c,d) : a, d \in \partial G\,\}. \tag{8.33}$$

It is clear that m is symmetric, that is,

$$m(a,b,c,d) = m(a,c,b,d) = m(b,a,d,c) \tag{8.34}$$

and also $\mathcal{GM}(\overline{\mathbf{R}}^n)$-invariant, that is,

$$m_f(a,b,c,d) = m(fa,fb,fc,fd) = m(a,b,c,d) \tag{8.35}$$

for all $f \in \mathcal{GM}(\overline{\mathbf{R}}^n)$ (cf. (1.28)). For $x, y \in \mathbf{R}^n \setminus \{a\}$ ($a \in \mathbf{R}^n$)

$$m(a,x,y,\infty) = \frac{|x-y|}{\min\{\, |x-a|,\ |y-a|\,\}}. \tag{8.36}$$

It follows from (8.36) that

$$j_G(x,y) = \log(1 + m_G(x,y)),\ G = \mathbf{R}^n \setminus \{a\}, \tag{8.37}$$

for all $x, y \in G$ where j_G is as in (2.34).

8.38. The point–pair invariant m_G . Next let us consider the conformally invariant symmetric function m_G for an arbitrary domain $G \subset \overline{\mathbf{R}}^n$ with $\mathrm{card}(\overline{\mathbf{R}}^n \setminus G) \geq 2$. The following properties are immediate:

(1) $G_1 \subset G_2$ and $x, y \in G_1 \implies m_{G_1}(x,y) \geq m_{G_2}(x,y)$.

(2) For a fixed $y \in G$, $m_G(x,y) \to 0$ iff $x \to y$ and $m_G(x,y) \to \infty$ iff $x \to \partial G$.

(3) $m_G(x,y) \geq q(\partial G)\, q(x,y)$.

(4) $m_G(x,y) \leq q(\partial G)\, q(x,y)/q(\{x,y\},\partial G)^2$.

8.39. Lemma. *$\rho(b,c) = \log(1 + m_{\mathbf{B}^n}(b,c))$ for $b, c \in \mathbf{B}^n$.*

Proof. By $\mathcal{GM}(\mathbf{B}^n)$–invariance we may assume $b = -re_1 = -c$. Then $\rho(b,c) = 2\log[(1+r)/(1-r)]$ or, equivalently, $r = \mathrm{th}\,\frac{1}{4}\rho(b,c)$. For all $a, d \in \partial\mathbf{B}^n$

$$m(a,b,c,d) \leq \frac{2|b-c|}{(1-r)^2} = \frac{4\,r}{(1-r)^2} = m(-e_1, -re_1, re_1, e_1)$$

and hence

$$m_{\mathbf{B}^n}(b,c) = \frac{4\,\mathrm{th}\,\frac{1}{4}\rho(b,c)}{(1-\mathrm{th}\,\frac{1}{4}\rho(b,c))^2} = e^{\rho(b,c)} - 1 \,. \quad \square$$

The similarities between $m(a,b,c,d)$ and $s(a,b,c,d)$ (see (3.21)) should be observed. In fact one could use $s(a,b,c,d)$ instead of $m(a,b,c,d)$ and prove analogous estimates.

Let a and d be distinct points in $\overline{\mathbf{R}}^n$ and $D = \overline{\mathbf{R}}^n \setminus \{a,d\}$.

8.40. Theorem. *$1 \leq \lambda_D(b,c)/\tau(m_D(b,c)) \leq 4$ for distinct $b, c \in D = \overline{\mathbf{R}}^n \setminus \{a,d\}$.*

Proof. The proof follows readily from (8.35) and 8.24. $\square$

8.41. Corollary. *Let $D \subset \overline{\mathbf{R}}^n$ be a c-QED domain with $\mathrm{card}(\overline{\mathbf{R}}^n \setminus D) \geq 2$. Then for distinct $x, y \in D$*

$$2^{1-n} c\,\tau(m) \leq c\,\tau(m^2 + 2m) \leq \lambda_D(x,y) \leq 4\,\tau(m)$$

where $m = m_D(x,y)$.

Proof. The proof follows from 8.29, 8.25, and 8.40. $\square$

8.42. Exercise. Let $b, c \in \mathbf{B}^n$. Show that

$$\rho(b,c) \le \log\Big(1 + \frac{2\,|b-c|}{(\min\{1-|b|, 1-|c|\})^2}\Big)\,,$$

$$\rho(b,c) \ge \log\Big(1 + \frac{2\,|b-c|}{\min\{1-|b|, 1-|c|\}\,(1+\min\{1-|b|, 1-|c|\})}\Big)\,.$$

Let $d \in S^{n-1}$. Show that

$$\rho(b,c) \ge \log\Big(1 + \frac{2\,|b-c|}{|b-d|\,|c+d|}\Big)\,.$$

8.43. Remarks. For $n = 2$ an explicit expression for the function $p(x)$ can be deduced from [KU, Theorem 5.2, p. 192]. This explicit expression is a real number determined by certain elliptic integrals with a complex argument. Because of this fairly complicated definition it is difficult to see how the exact value of $p(x)$ changes with x or, say, with the angles α between $[0,x]$ and $[0,e_1]$ and β between $[0,e_1]$ and $[e_1,x]$, respectively. In [VU13, 4.3] it was conjectured that for $n = 2$ the constant 4 in 8.24 can be replaced by a smaller one, $c = 1.1712\ldots = \mu(1/\sqrt{3})/\mu(1/\sqrt{2})$, which would be sharp as shown in [VU13, 4.3]. A weaker version of this conjectured two–dimensional result with c^2 in place of c was established in [LEVU]. Also some bounds for $\lambda_{\mathbf{B}^2\setminus\{0\}}(x,y)$ were found in [LEVU].

8.44. Exercise. Mori's ring $R_{M,2}(\alpha,\beta)$ in $\mathbf{R}^2$ has two complementary components $C_1 = \{\, te_1 : t \ge 0 \,\}$ and $C_0 = \{\, (\cos\varphi, \sin\varphi) \in \mathbf{R}^2 : \pi - \alpha \le \varphi \le \pi + \beta \,\}$, $0 < \alpha \le \beta < \pi$. Find an expression for $\operatorname{cap} R_{M,2}(\alpha,\beta)$ by mapping $\mathbf{R}^2 \setminus C_1$ conformally onto $\mathbf{H}^2$. (For $n = 2$ $p((\frac{1}{2}, y))$ can be expressed in terms of the capacity of Mori's ring, see [KU, Theorem 5.2, p. 192].)

8.45. Remark. One can show that $\lambda_{\mathbf{B}^n}(x,y)^{1/(1-n)}$ is a metric on $\mathbf{B}^n$ [AVV3]. It is tempting to conjecture that for all proper subdomains G of $\mathbf{R}^n$, $\lambda_G(x,y)^{1/(1-n)}$ is a metric. Even the particular case $n = 2$, $G = \mathbf{R}^2 \setminus \{0\}$, is open. As shown in [LF2] $\lambda_G(x,y)^{-1/n}$ is always a metric.

Next we shall find an upper bound for the function $\alpha_{K,n}(t)$ defined as

$$\alpha_{K,n}(t) = \tau_n^{-1}\big(\tau_n(t)/K\big)\,, \quad t > 0\,, \quad K > 0\,. \tag{8.46}$$

It is easy to show using the basic functional identity 5.53 that

$$\alpha_{K,n}(t) = \frac{1-B^2}{B^2}\;; \quad B = \varphi_{1/K,n}\big(1/\sqrt{1+t}\,\big)\,.$$

For $n = 2$ we can go one step further using the identity 5.61(2) and obtaining (7.54) as a result. Further from (7.54) one can easily deduce that $\alpha_{K,2}(t)$ has a majorant of the form $A\,t^{1/K}$, A constant as we have pointed out earlier. Although the multidimensional analogue of 5.61(2) is false (recall 7.58), we nevertheless can find a similar majorant for $\alpha_{K,n}(t)$ valid for all dimensions $n \geq 2$.

8.47. Theorem. *For $n \geq 2$, $K \geq 1$, and $t \in (0, 2^{2-3K})$ the following inequality holds*

$$\tau_n^{-1}(\tau_n(t)/K) \leq 4^{3-1/K} t^{1/K} .$$

Proof. Let $x = \tau_n^{-1}(\tau_n(t)/K)$ and $b = \log(1 + 2(1 + \sqrt{1+t})/t)$. By 7.26(3) we obtain

$$c_n b \leq \tau_n(t) = K\tau_n(x) \leq c_n K \mu(1 + 2(1 - \sqrt{1+x})/x)$$

and further

$$x \leq \frac{4\,\mu^{-1}(b/K)}{(1 - \mu^{-1}(b/K))^2} .$$

The inequality $\log(1/r) < \mu(r) < \log(4/r)$ (cf. (5.58)) shows that $e^{-u} < \mu^{-1}(u) < 4e^{-u}$ for $u > 0$. Therefore $\mu^{-1}(b/K) < \frac{1}{2}$ for $t \in (0, 2^{2-3K})$ and also

$$x \leq 4^3 \left(\frac{t}{t + 2(1 + \sqrt{1+t})} \right)^{1/K} \leq 4^{3-1/K} t^{1/K}$$

holds for $t \in (0, 2^{2-3K})$ as desired. □

8.48. Notes. This section is taken from [VU10] and [VU13]. NED sets in the complex plane were introduced by L. V. Ahlfors and A. Beurling [AB]. J. Väisälä [V3] studied NED sets in n-space and finally F. W. Gehring and O. Martio [GM1] introduced QED sets. See also V. V. Aseev and A. V. Sychev [ASY] as well as J. Väisälä [V12]. The conformal metric μ_G has been studied by I. S. Gál [GÁL] and T. Kuusalo [K1]. Its dual invariant λ_G was introduced by J. Ferrand [LF2]. For $n = 2$ the function $p(x)$ is closely connected to a modulus problem of O. Teichmüller and the shape of the extremal ring for $p(x)$ has been thoroughly examined (see G. V. Kuz'mina [KU, p. 192, Theorem 5.2]).

8.49. An appendix to Section 8. We shall give here an alternative proof for Theorem 8.12(1) which is slightly simpler than the proof given in 8.17. This proof is due to M. K. Vamanamurthy. First a lemma is needed.

8.50. Lemma. *If* $t > 0$ *and* $s = t^2/(1+\sqrt{1+t^2})$ *, then the inequality* $2\,s(1+\sqrt{1+1/s}) \ge t\sqrt{2}$ *holds.*

Proof. The assertion is equivalent to

$$\frac{\sqrt{2}\,t^2}{1+\sqrt{1+t^2}}\left(1+\sqrt{1+\frac{1+\sqrt{1+t^2}}{t^2}}\right) \ge t\,,$$

or to

$$\frac{\sqrt{2}\,(t+\sqrt{t^2+1+\sqrt{1+t^2}}\,)}{1+\sqrt{1+t^2}} \ge 1\,.$$

This is equivalent to

$$f(t) = \frac{\sqrt{2}\,(t+\sqrt[4]{1+t^2}\,\sqrt{1+\sqrt{1+t^2}}\,)}{1+\sqrt{1+t^2}} \ge 1\,.$$

But here the left side

$$f(t) \ge \frac{\sqrt{2}\,\sqrt[4]{1+t^2}}{\sqrt{1+\sqrt{1+t^2}}} = \frac{\sqrt{2}\,u}{\sqrt{1+u^2}} \ge 1$$

since $u/\sqrt{1+u^2}$ is increasing on $[1,\infty)$ and $u = \sqrt[4]{1+t^2} \ge 1$. ❐

8.51. A second proof for Theorem 8.12(1). For $x \in D_1 \setminus B^n(-e_1,2)$ we have $x = x+e_1-e_1$ and $x-e_1 = x+e_1-2e_1$. Hence

$$|x|^2 = |x+e_1|^2+1-2(x+e_1)\cdot e_1\,,$$
$$|x-e_1|^2 = |x+e_1|^2+4-4(x+e_1)\cdot e_1\,.$$

These inequalities yield $2|x|^2-|x-e_1|^2 = |x+e_1|^2-2 \ge 2$. Thus $|x|^2-1 \ge \frac{1}{2}|x-e_1|^2 = t^2$ and hence

$$|x|-1 = \frac{|x|^2-1}{|x|+1} \ge \frac{\frac{1}{2}|x-e_1|^2}{1+\sqrt{1+\frac{1}{2}|x-e_1|^2}} = \frac{t^2}{1+\sqrt{1+t^2}} = s\,.$$

By 5.63(2) and 8.50

$$\tau(|x|-1) \le 2\,\tau\big(2s(1+\sqrt{1+1/s})\big) \le 2\,\tau(t\sqrt{2}\,) = 2\,\tau(|x-e_1|)$$

as desired. ❐

Chapter III
QUASIREGULAR MAPPINGS

The study of quasiconformal and quasiregular mappings in this and the following chapter will be based on the transformation formulae for the moduli of curve families under these mappings. In most cases it will be enough to make use of these transformation formulae specialized to the conformal invariants μ_G and λ_G. These special cases of the general transformation formulae are convenient to use because they together with the results of Section 8 provide immediate insight into some relevant geometric quantities.

In the case of the conformal (pseudo)metric μ_G the transformation formula reads: a quasiregular mapping $f\colon G \to fG \subset \mathbf{R}^n$ is a Lipschitz mapping between the (pseudo)metric spaces (G, μ_G) and (fG, μ_{fG}). From this result and a similar result for the conformal invariant λ_G we derive several distortion and growth theorems for quasiregular mappings.

To this end we shall make use of some results from Chapter II that will enable us to find simple estimates for the functions $\mu_G(x,y)$ and $\lambda_G(x,y)$. Except for the special case $G = \mathbf{B}^n$ formulae for $\mu_G(x,y)$ and $\lambda_G(x,y)$ are unknown, but one can give upper and lower bounds for them in terms of

$$r_G(x,y) = \frac{|x-y|}{\min\{d(x),\, d(y)\}}\ , \quad d(x) = d(x, \partial G)\ ,$$

for a wide class of domains G (see 3.8 and 8.26).

When $G = \mathbf{B}^n$ the transformation formulae for μ_G and λ_G yield two variants of the Schwarz lemma (see 11.2 and 11.22, respectively). A central theme of this chapter is a circle of ideas centered in the Schwarz lemma and its various generalizations, including a study of uniform continuity properties of qr mappings. In particular, we shall also discuss some properties of normal quasiregular mappings.

9. Topological properties of discrete open mappings

In this section we shall survey some topological properties of discrete open mappings. A thorough discussion of this topic, including the definition of the degree of a mapping, requires machinery from algebraic topology (see [RR]). In this section no proofs will be given.

9.1. Definition. The set $\mathbf{T}^n$ consists of all triples (y, f, D), where $f\colon G \to \overline{\mathbf{R}}^n$ is a continuous mapping, $G \subset \overline{\mathbf{R}}^n$ is a domain, D is a domain with $\overline{D} \subset G$ and $y \in \overline{\mathbf{R}}^n \setminus f\partial D$.

9.2. Lemma. *There exists a unique function* $\mu : \mathbf{T}^n \to \mathbf{Z}$, *the topological degree, such that*

(1) $y \mapsto \mu(y, f, D)$ *is a constant in each component of* $\overline{\mathbf{R}}^n \setminus f\partial D$.

(2) $|\mu(y, f, D)| = 1$ *if* $y \in fD$ *and* $f|\overline{D}$ *is one-to-one.*

(3) $\mu(y, \mathrm{id}, D) = 1$ *if* $y \in D$ *and* id *is the identity mapping.*

(4) *Let* $(y, f, D) \in \mathbf{T}^n$ *and* $D_1, \ldots, D_k$ *be disjoint domains such that* $(y, f, D_i) \in \mathbf{T}^n$ *and* $f^{-1}(y) \cap D \subset \bigcup_{i=1}^{k} D_i$. *Then*

$$\mu(y, f, D) = \sum_{i=1}^{k} \mu(y, f, D_i) .$$

(5) *Let* $(y, f, D), (y, g, D) \in \mathbf{T}^n$ *be such that there exists a homotopy* $h_t\colon \overline{D} \to \overline{\mathbf{R}}^n$, $t \in [0,1]$, *with* $h_0 = f|\overline{D}$, $h_1 = g|\overline{D}$, *and* $(y, h_t, D) \in \mathbf{T}^n$ *for all* $t \in [0,1]$. *Then* $\mu(y, f, D) = \mu(y, g, D)$.

9.3. Lemma. (1) *If* $(y, f, D) \in \mathbf{T}^n$ *and* $y \notin f\overline{D}$, *then* $\mu(y, f, D) = 0$.

(2) *If* f *is a constant* c, *then* $\mu(y, f, D) = 0$ *for all* $y \neq c$.

(3) *If* $f\colon D \to \mathbf{R}^n$ *is differentiable at* $x_0 \in D$ *and* $J_f(x_0) = \det f'(x_0) \neq 0$, *then there exists a neighborhood* U *of* x_0 *such that* $(y, f, U) \in \mathbf{T}^n$ *and* $\mu(y, f, U) = \operatorname{sign} J_f(x_0)$ *for* $y \in fU$.

It follows from 9.3(3) that if f is a reflection in the plane $x_n = 0$, then $\mu(y, f, \mathbf{B}^n) = -1$ for $y \in \mathbf{B}^n$. We next extend the definition 1.7 of a sense-preserving $\mathcal{C}^1$-homeomorphism.

9.4. Definition. A mapping $f\colon G \to \overline{\mathbf{R}}^n$ is called *sense-preserving (orientation-preserving)* if $\mu(y, f, D) > 0$ whenever D is a domain with $\overline{D} \subset G$ and $y \in fD \setminus f\partial D$. If $\mu(y, f, D) < 0$ for all such y and D, then f is called *sense-reversing (orientation-reversing).*

Reflection in a plane and inversion in a sphere are sense-reversing mappings ([RR, pp. 137–145]).

9.5. Lemma. *Let $f\colon G \to \overline{\mathbf{R}}^n$ and $g\colon fG \to \overline{\mathbf{R}}^n$ be mappings and set $h = g \circ f$. If f and g are both sense-preserving or both sense-reversing, then h is sense-preserving. If one of the maps f and g is sense-reversing and the second is sense-preserving, then h is sense-reversing.*

9.6. Remarks. The approach to the degree theory in [RR] is based on algebraic topology. An alternative approach can be based on Sard's theorem and on approximation of continuous functions by $\mathcal{C}^\infty$-functions, for which the degree $\mu(y, f, D)$ can be defined as the sum of the signs of the Jacobians, evaluated at the points of $D \cap f^{-1}(y)$. See [DE], [HEI], [R12].

9.7. Lemma. *Let (y, f, D) and $(y, g, D) \in \mathbf{T}^n$ be such that $f|\partial D = g|\partial D$ and $\infty \notin f\overline{D} \cup g\overline{D}$. Then $\mu(y, f, D) = \mu(y, g, D)$.*

For 9.5 see [V4] and for 9.7 see [RR, pp. 129–130]. The assumption $\infty \notin f\overline{D} \cup g\overline{D}$ in 9.7 cannot be dropped, as the example $D = \mathbf{B}^n$, $f = \mathrm{id}$, and g an inversion in S^{n-1}, shows.

The *branch set* B_f of a mapping $f\colon G \to \overline{\mathbf{R}}^n$ is defined to be the set of all points $x \in G$ such that f is not a local homeomorphism at x. It is easily seen that B_f is a closed subset of G. We call f *open* if fA is open in $\overline{\mathbf{R}}^n$ whenever $A \subset G$ is open, *light* if $f^{-1}(y)$ is totally disconnected for all $y \in fG$, and *discrete* if $f^{-1}(y)$ is isolated for all $y \in fG$.

The next lemma is a fundamental property of discrete open mappings (see A. V. Chernavskiĭ [CHE1], [CHE2] and J. Väisälä [V5]).

9.8. Lemma. *Let* $f\colon G \to \overline{\mathbf{R}}^n$ *be discrete open. Then* $\dim B_f = \dim fB_f = \dim f^{-1}fB_f \le n-2$, *where* dim *refers to the topological dimension.*

9.9. Remarks. It is clear that $G \setminus B_f$ is open, and from 9.8 and a well-known non-separation property of sets of dimension $\le n-2$ (see [HW, p. 98]) it follows that $G \setminus B_f$ is a domain. Stoilow's theorem (see [LV2]) implies that B_f consists of isolated points for $n = 2$. In the multidimensional case $n \ge 3$ B_f never contains isolated points, as one can show by applying some properties of covering mappings (monodromy theorem).

Let $G \subset \overline{\mathbf{R}}^n$ be a domain. We denote by $J(G)$ the collection of all subdomains D of G with $\overline{D} \subset G$.

9.10. Definition. Let $f\colon G \to \overline{\mathbf{R}}^n$ be discrete. Fix $x \in G$ and a neighborhood $U \in J(G)$ of x such that $\{x\} = \overline{U} \cap f^{-1}(f(x))$. The number $\mu(f(x), f, U)$ is denoted by $i(x,f)$ and called the *local* (topological) *index* of f at x. (Exercise. Making use of 9.2(4) show that $i(x,f)$ is independent of the neighborhood of x and has the required properties.)

Now let $f\colon G \to \mathbf{R}^n$ be discrete open. It follows from 9.8 that $G \setminus B_f$ is connected. Hence $i(x,f)$ has a constant value, either $+1$ or -1, in $G \setminus B_f$. In the first case f is sense-preserving, and in the second case sense-reversing. In both cases we have by 9.2(4) if $D \in J(G)$, $y \in fD \setminus f\partial D$, and $D \cap f^{-1}(y) = \{x_1, \ldots, x_k\}$

$$\mu(y,f,D) = \sum_{j=1}^{k} i(x_j, f)\ . \tag{9.11}$$

A domain $D \in J(G)$ is said to be a *normal domain* of $f\colon G \to \mathbf{R}^n$ if $f\partial D = \partial fD$. A *normal neighborhood* of x is a normal domain D such that $D \cap f^{-1}(f(x)) = \{x\}$.

It follows from 9.2(1) that $\mu(y,f,D)$ is a constant if D is a normal domain of f and $y \in fD$. This constant is denoted $\mu(f,D)$. Let D be a normal domain of f, $y \in fD$, and $f^{-1}(y) = \{x_1, \ldots, x_k\}$. It follows from (9.10) that

$$\mu(f,D) = \sum_{j=1}^{k} i(x_j, f)\ .$$

9.12. Exercise. If $f\colon G \to \mathbf{R}^n$ is open and $D \in J(G)$, then $\partial fD \subset f\partial D$ is always true.

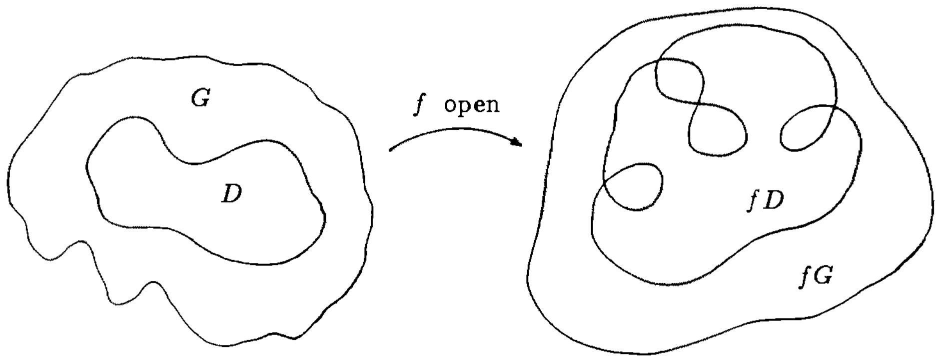


Diagram 9.1.

In classical function theory (see [BU, p. 84], [WH2]) the local topological index is usually called the winding number of a point.

We shall next list several topological results about discrete open mappings without proofs. The proofs of Lemmas 9.13–9.15 are given in [MRV1].

9.13. Lemma. *Suppose that* $f\colon G \to \mathbf{R}^n$ *is open, that* $U \subset \mathbf{R}^n$ *is a domain, and that* D *is a component of* $f^{-1}U$ *such that* $D \in J(G)$ *. Then* D *is a normal domain,* $fD = U$ *, and* $U \in J(fG)$ *.*

If $f\colon G \to \mathbf{R}^n$, $x \in G$, and $r > 0$, then the x–component of $f^{-1}B^n(f(x), r)$ is denoted by $U(x, f, r)$.

9.14. Lemma. *Suppose that* $f\colon G \to \mathbf{R}^n$ *is a discrete and open mapping. Then* $\lim_{r\to 0} d(U(x, f, r)) = 0$ *for every* $x \in G$ *. If* $U(x, f, r) \in J(G)$ *, then* $U(x, f, r)$ *is a normal domain and* $fU(x, f, r) = B^n(f(x), r) \in J(fG)$ *. Furthermore, for every point* $x \in G$ *there is a positive number* σ_x *such that the following conditions are satisfied for* $0 < r \leq \sigma_x$ *:*

(1) $U(x, f, r)$ *is a normal neighborhood of* x *.*

(2) $U(x, f, r) = U(x, f, \sigma_x) \cap f^{-1}B^n(f(x), r)$ *.*

(3) $\partial U(x, f, r) = U(x, f, \sigma_x) \cap f^{-1}S^{n-1}(f(x), r)$ *if* $r < \sigma_x$ *.*

(4) $\overline{\mathbf{R}}^n \setminus U(x, f, r)$ *is connected.*

(5) $\overline{\mathbf{R}}^n \setminus \overline{U}(x, f, r)$ *is connected.*

(6) *If* $0 < r < s \leq \sigma_x$ *, then* $\overline{U}(x, f, r) \subset U(x, f, s)$ *, and* $U(x, f, s) \setminus \overline{U}(x, f, r)$ *is a ring, i.e. its complement has exactly two components.*

If $f: G \to \mathbf{R}^n$, $A \subset \mathbf{R}^n$ and $y \in \mathbf{R}^n$, denote

$$\begin{aligned} N(y,f,A) &= \operatorname{card}(A \cap f^{-1}(y)) , \\ N(f,A) &= \sup\{ N(y,f,A) \, : \, y \in \mathbf{R}^n \} , \\ N(f) &= N(f,G) . \end{aligned}$$

Here $N(y,f,A)$ is called the *multiplicity* of y in A and $N(f,A)$ the *maximal multiplicity* of f in A.

9.15. Lemma. *Suppose that $f: G \to \mathbf{R}^n$ is sense-preserving, discrete, and open.*

(1) *If $D \in J(G)$, then $N(y,f,D) \le \mu(y,f,D)$ for all $y \in \mathbf{R}^n \setminus f\partial D$ and $N(y,f,D) = \mu(y,f,D)$ for $y \in \mathbf{R}^n \setminus fA$, $A = \partial D \cup (D \cap B_f)$.*

(2) *If D is a normal domain, $N(f,D) = \mu(f,D)$.*

(3) *If $A \subset G$ is compact, $N(f,A) < \infty$.*

(4) *Every point $x \in G$ has a neighborhood V such that if U is a neighborhood of x and if $U \subset V$, then $N(f,U) = i(x,f)$.*

(5) *$x \in B_f$ iff $i(x,f) \ge 2$.*

It follows from 9.15(4) that the local index $i(x,f)$ of a sense-preserving discrete open mapping f can be defined in terms of the maximal multiplicity of f as follows

$$i(x,f) = \lim_{r \to 0} N(f, B^n(x,r)) . \tag{9.16}$$

A trivial example is the function $g: \mathbf{B}^2 \to \mathbf{B}^2$, $g(z) = z^2$ with $i(0,g) = 2$.

9.17. Remark. Let $f: G \to \mathbf{R}^n$ be continuous, $A_j \subset \mathbf{R}^n$, $j = 1,2,\ldots$. Then one can show that

$$\begin{aligned} N(y,f,\textstyle\bigcup A_j) &\le \textstyle\sum N(y,f,A_j) , \\ N(f,\textstyle\bigcup A_j) &\le \textstyle\sum N(f,A_j) . \end{aligned}$$

If A is a Borel set in G, then $N(y,f,A)$ is measurable (cf. [RR, pp. 216–219]).

9.18. An open problem. Let $f: G \to \mathbf{R}^n$ be discrete open, $x_0 \in G$, $t \in (0, d(x_0,\partial G))$, and assume that $fS^{n-1}(x_0,t) = \partial f B^n(x_0,t)$, that is, $B^n(x_0,t)$ is a normal domain. Assume, further, that $B_f \cap S^{n-1}(x_0,t) = \emptyset$ and $n \ge 3$. Is it true that $f|B^n(x_0,t)$ is one-to-one? For $n = 2$ we have the obvious counterexample $g: \mathbf{B}^2 \to \mathbf{B}^2$, $g(z) = z^2$. This problem is given in [BBH, p. 503, 7.66].

9.19. Path lifting. Let $f\colon G \to \overline{\mathbf{R}}^n$ and let $\beta\colon [a,b) \to \overline{\mathbf{R}}^n$ be a path and let $x_0 \in G$ be such that $f(x_0) = \beta(a)$. A path $\alpha\colon [a,c) \to G$ is said to be a *maximal lifting* of β *starting* at x_0 if:

(1) $\alpha(a) = x_0$.

(2) $f \circ \alpha = \beta|[a,c)$.

(3) If $c < c' \leq b$, then there does not exist a path $\alpha'\colon [a,c') \to G$ such that $\alpha = \alpha'|[a,c)$ and $f \circ \alpha' = \beta|[a,c')$.

If $\beta\colon [a,b) \to \overline{\mathbf{R}}^n$ is a path and if $C \subset \overline{\mathbf{R}}^n$, we write $\beta(t) \to C$ as $t \to b$ if the spherical distance $q(\beta(t), C) \to 0$ as $t \to b$.

9.20. Lemma. *Suppose that $f\colon G \to \overline{\mathbf{R}}^n$ is light and open, that $x_0 \in G$, and that $\beta\colon [a,b) \to \overline{\mathbf{R}}^n$ is a path such that $\beta(a) = f(x_0)$ and such that either $\lim_{t\to b} \beta(t)$ exists or $\beta(t) \to \partial fG$ as $t \to b$. Then β has a maximal lifting $\alpha\colon [a,c) \to G$ starting at x_0. If $\alpha(t) \to x_1 \in G$ as $t \to c$, then $c = b$ and $f(x_1) = \lim_{t\to b} \beta(t)$. Otherwise $\alpha(t) \to \partial G$ as $t \to c$. If f is discrete and if the local index $i(\alpha(t), f)$ is constant for $t \in [a,c)$, then α is the only maximal lifting of β starting at x_0.*

This lemma is proved in [MRV3, 3.12].

It follows from the lemma, in particular, that a locally homeomorphic mapping has a unique maximal lifting starting at a point.

9.21. Remarks. In the sequel Lemma 9.20 will be applied in the following situation. Let $f\colon G \to \mathbf{R}^n$ be non-constant qr, $x_0 \in G$, and let $\beta\colon [0,1] \to \overline{\mathbf{R}}^n$ be a path with $\beta(0) = f(x_0)$ and $\beta(1) \in \partial fG$. Then 9.20 shows that β has a maximal lifting $\alpha\colon [0,c) \to G$ starting at x_0 with $\alpha(t) \to \partial G$ as $t \to c$.

A mapping $f\colon G \to \mathbf{R}^n$ is called *proper* if $f^{-1}K$ is a compact subset of G whenever K is a compact subset of fG, and *closed* if fC is a (relatively) closed subset of fG whenever C is a (relatively) closed subset of G.

9.22. Lemma. *Let $f\colon G \to \mathbf{R}^n$ be discrete open. Then the following conditions are equivalent:*

(1) *f is proper.*

(2) *f is closed.*

(3) *$N(f,G) = p < \infty$ and for all $y \in fG$*

$$p = \sum_{j=1}^{k} i(x_j, f)\ , \quad \{x_1, \ldots, x_k\} = f^{-1}(y)\ .$$

For the proof of 9.22 see [V5], [MSR1], [VU1], and the references in these papers. As the simple example $z \mapsto z^2$ shows, a maximal lifting of a path starting at a branch point need not be unique. The next lemma is a quantitative statement of this fact. For a proof see [RI2].

9.23. Lemma. *Let $f\colon G \to \mathbf{R}^n$ be discrete, open, and closed. Denote $p = N(f,G) < \infty$ and let $\beta\colon [a,b) \to fG$ be a path. Then there exist paths $\alpha_j\colon [a,b) \to G$, $1 \le j \le p$, for which*

(1) $f \circ \alpha_j = \beta$,

(2) $\operatorname{card}\{\, j : \alpha_j(t) = x \,\} = |i(x,f)|$ *for* $x \in f^{-1}|\beta|$ *and* $t \in [a,b)$,

(3) $\bigcup_{j=1}^{p} |\alpha_j| = f^{-1}|\beta|$.

9.24. Remarks. It follows easily from the definitions that an open continuous mapping $f\colon G \to \mathbf{R}^n$ obeys the *maximum principle*, i.e. if $D \in J(G)$ then

$$\max_{\partial D} |f(x)| = \max_{D} |f(x)| .$$

For further results concerning with discrete and open mappings see [CH] and [TY].

10. Some properties of quasiregular mappings

In the present section we study some fundamental properties of quasiregular mappings. According to deep results of Yu. G. Reshetnyak [R2], [R12], a non-constant quasiregular mapping is discrete, open, and differentiable a.e., and it satisfies Lusin's condition (N) [HS, p. 288]. By definition, condition (N) holds if and only if sets of measure zero are mapped onto sets of measure zero. The proofs are beyond the scope of this book. Applying these results one can prove the transformation formulae, the so-called K_O- and K_I-inequalities, for the moduli of curve families under quasiregular mappings. Also these important results are stated without proof. Of these the K_O-inequality is due to O. Martio, S. Rickman, and J. Väisälä [MRV1], while the K_I-inequality was proved by E. A. Poletskiĭ [P1] and in an improved form by J. Väisälä [V8]. A simplified proof of Poletskiĭ's result was given by M. Pesonen [PE2].

10.1. Quasiregular mappings. Let $G \subset \mathbf{R}^n$ be a domain. A mapping $f\colon G \to \mathbf{R}^n$ is said to be *quasiregular* (qr) if f is ACL^n and if there exists a constant

$K \geq 1$ such that

$$(10.2)\qquad |f'(x)|^n \leq K J_f(x)\ ,\quad |f'(x)| = \max_{|h|=1} |f'(x)h|\ ,$$

a.e. in G. Here $f'(x)$ denotes the formal derivative of f at x (cf. Notation and terminology). The smallest $K \geq 1$ for which this inequality is true is called the *outer dilatation* of f and denoted by $K_O(f)$. If f is quasiregular, then the smallest $K \geq 1$ for which the inequality

$$(10.3)\qquad J_f(x) \leq K\, l(f'(x))^n\ ,\quad l(f'(x)) = \min_{|h|=1} |f'(x)h|\ ,$$

holds a.e. in G is called the *inner dilatation* of f and denoted by $K_I(f)$. The *maximal dilatation* of f is the number $K(f) = \max\{ K_I(f), K_O(f) \}$. If $K(f) \leq K$, f is said to be K*-quasiregular* (K-qr). If f is not quasiregular, we set $K_O(f) = K_I(f) = K(f) = \infty$.

It follows from linear algebra (see [V7, p. 44] and [R12, p. 22]) that

$$(10.4)\qquad K_O(f) \leq K_I(f)^{n-1}\ ,\quad K_I(f) \leq K_O(f)^{n-1}$$

hold. Moreover, these inequalities are best possible.

10.5. Lemma. *Let $f\colon G \to \mathbf{R}^n$ be a non-constant qr mapping. Then*

(1) *f is sense-preserving, discrete, and open,*

(2) *f is differentiable a.e.,*

(3) *f satisfies condition (N), i.e. if $A \subset G$ and $m(A) = 0$, then also $m(fA) = 0$.*

For proofs of these results, see [R2], [R4], [R12].

Next we extend the definition of a qr mapping.

10.6. Quasimeromorphic mappings. Let $G \subset \overline{\mathbf{R}}^n$ be a domain. A mapping $f\colon G \to \overline{\mathbf{R}}^n$ is called *quasimeromorphic* (qm) if either $fG = \{\infty\}$ or the set $E = f^{-1}(\infty)$ is discrete and $f_1 = f | G \setminus (E \cup \{\infty\})$ is qr. We set $K(f) = K(f_1)$, $K_O(f) = K_O(f_1)$, and $K_I(f) = K_I(f_1)$.

10.7. Quasiconformal mappings. If f is a homeomorphism satisfying (10.2) and (10.3) with $|J_f(x)|$ in place of $J_f(x)$, then f is called *quasiconformal* (qc).

10.8. Remarks. For $n = 2$ and $K = 1$ the class of K-qr maps coincides with the class of analytic functions. By 10.7 a qc mapping may be sense-reversing, while a qr mapping in the sense of (10.2) is always sense-preserving. In his book [R12] Reshetnyak replaces $J_f(x)$ by $|J_f(x)|$ in the definition of a qr mapping and hence qr maps in the sense of [R12] may be sense-reversing. This is largely a question of technical convention, since by topology (see Lemma 9.8) each discrete open mapping is either sense-preserving or sense-reversing.

10.9. Curve families and quasiconformal mappings. We now give an alternative definition of a quasiconformal mapping. Let G, G' be domains in $\overline{\mathbf{R}}^n$ and let $f\colon G \to G'$ be a homeomorphism. Then f is *K-quasiconformal* if

$$\mathsf{M}(\Gamma)/K \le \mathsf{M}(f\Gamma) \le K\,\mathsf{M}(\Gamma) \tag{10.10}$$

for every curve family Γ in G. Moreover, the dilatations of f are defined as

$$K_I(f) = \sup \frac{\mathsf{M}(f\Gamma)}{\mathsf{M}(\Gamma)} \;, \quad K_O(f) = \sup \frac{\mathsf{M}(\Gamma)}{\mathsf{M}(f\Gamma)} \;,$$

where the suprema are taken over all curve families Γ in G such that $\mathsf{M}(\Gamma)$ and $\mathsf{M}(f\Gamma)$ are not simultaneously 0 or ∞. Thus

$$\mathsf{M}(\Gamma)/K_O(f) \le \mathsf{M}(f\Gamma) \le K_I(f)\,\mathsf{M}(\Gamma) \tag{10.11}$$

for every curve family Γ in G. The equivalence of the two definitions 10.7 and (10.10) of a qc mapping is proved in [V7] and also in [C1, pp. 81–110].

The next example shows that (10.11) does not generalize directly to the case of qr mappings.

10.12. Examples. (1) Let $f_k(z) = z^k$, $k \in \mathbf{N} \setminus \{0\}$, $z \in \mathbf{C} = \mathbf{R}^2$, and $\Gamma = \Delta(S^1, S^1(1/e))$. By (5.14)

$$\mathsf{M}(\Gamma) = 2\pi \;, \quad \mathsf{M}(f_k\Gamma) \le 2\pi/\log(e^k) = 2\pi/k \;.$$

Moreover, $K(f_k) = 1$ because f_k is analytic. If $k \ge 2$, we see that the left inequality of (10.11) fails to hold for (non-univalent) analytic functions.

(2) Let $A_j = \{\, (x,y) \in \mathbf{R}^2 : x = j \,\}$, $j = 0,1$, $f(z) = \exp z$, $z \in \mathbf{R}^2$, and $\Gamma = \Delta(A_0, A_1)$. Then $f\Gamma \subset \Delta(S^1(e), S^1)$ and $\mathsf{M}(f\Gamma) \le 2\pi/\log e = 2\pi$ by (5.14), whereas $\mathsf{M}(\Gamma) = \infty$ by 5.11 or by 5.33 and 5.17. Since $K(f) = 1$ also in this example, we see that the left inequality of (10.11) fails to hold for analytic functions. A fortiori, it fails to hold for qr mappings.

By inserting a multiplicity factor in the left side of (10.11) one obtains the K_O-inequality for quasiregular mappings ([MRV1]).

10.13. Theorem. *Suppose that* $f\colon G \to \mathbf{R}^n$ *is a quasiregular mapping and that* A *is a Borel set in* G *such that* $N(f,A) < \infty$ *. If* Γ *is a family of paths in* A *,*

$$\mathsf{M}(\Gamma) \le N(f,A)\, K_O(f)\, \mathsf{M}(f\Gamma)\ .$$

Proof. Set

$$L(x,f) = \limsup_{h\to 0} \frac{|f(x+h)-f(x)|}{|h|}$$

for $x \in G$. Thus $L(x,f) = |f'(x)|$ whenever f is differentiable at x. It is easy to see that $x \mapsto L(x,f)$ is a Borel function.

Suppose that $\sigma \in \mathcal{F}(f\Gamma)$. Define $\rho\colon \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ by setting

$$\rho(x) = \begin{cases} \sigma(f(x))\, L(x,f) & \text{if } x \in A\,; \\ 0 & \text{otherwise}. \end{cases}$$

Let Γ_0 be the family of all rectifiable paths $\gamma \in \Gamma$ such that f is absolutely continuous on γ. By Lemma 7.5 $\mathsf{M}(\Gamma_0) = \mathsf{M}(\Gamma)$. From the formula concerning change of variables in integrals it follows that

$$\int_\gamma \rho\, ds \ge \int_{f\circ\gamma} \sigma\, ds \ge 1$$

for all $\gamma \in \Gamma_0$. Thus $\rho \in \mathcal{F}(\Gamma_0)$. A more detailed proof is given in [MRV1]. Hence we obtain

$$\begin{aligned} \mathsf{M}(\Gamma) = \mathsf{M}(\Gamma_0) &\le \int_{\mathbf{R}^n} \rho^n\, dm = \int_A \sigma(f(x))^n\, L(x,f)^n dm(x) \\ &\le K_O(f) \int_A \sigma(f(x))^n\, J(x,f)\, dm(x)\ . \end{aligned}$$

Since f is ACL^n, $J(x,f)$ is integrable over every domain $D \in J(G)$. Thus the transformation formula in [RR, Theorem 3, p. 364] yields

$$\begin{aligned} \int_{A\cap D} \sigma(f(x))^n\, J(x,f)\, dm(x) &= \int_{\mathbf{R}^n} \sigma(y)^n\, N(y,f,A\cap D)\, dm(y) \\ &\le N(f,A) \int_{\mathbf{R}^n} \sigma^n dm\ . \end{aligned}$$

The theorem cited above is formulated in [RR] for finite-valued functions, but we may apply it to $\min(k,\sigma^n)$ and then let $k \to \infty$. Since $D \in J(G)$ is arbitrary, we obtain

$$\mathsf{M}(\Gamma) \le N(f,A)\, K_O(f) \int_{\mathbf{R}^n} \sigma^n dm\ .$$

Since this holds for every $\sigma \in \mathcal{F}(f\Gamma)$, the theorem follows. ❒

The right side of (10.11) holds for qr mappings, too, as the following theorem shows. We shall mainly need the special case $m = 1$ of this result. The proof is omitted ([V8]).

10.14. Theorem. *Suppose that* $f\colon G \to \mathbf{R}^n$ *is a non-constant qr mapping, that* Γ *is a path family in* G, *that* Γ' *is a path family in* $\mathbf{R}^n$ *and that* m *is a positive integer such that the following condition is satisfied: There is a set* $E_0 \subset G$ *of measure zero such that for every path* $\beta\colon I \to \mathbf{R}^n$ *in* Γ' *there are paths* $\alpha_1, \ldots, \alpha_m$ *in* Γ *with* $f \circ \alpha_i \subset \beta$ *for all* i *and such that for every* $x \in G \setminus E_0$ *and* $t \in I$, $\alpha_i(t) = x$ *for at most one* i. *Then*

$$\mathsf{M}(\Gamma') \le \frac{K_I(f)}{m}\,\mathsf{M}(\Gamma)\,.$$

In this result it is not required that $f\Gamma = \Gamma'$. As a matter of fact, in many applications $f\Gamma < \Gamma'$. If D is a normal domain of f, if Γ' is a family of paths in fD, and if Γ is the family of all paths α in D such that $f \circ \alpha \in \Gamma'$, then the condition in 10.14 is satisfied with $m = N(f, D)$, $E_0 = B_f$ by 9.22 and 10.16(1) below.

Due to the connection (7.10) between the conformal capacity and the modulus of a curve family, one can formulate the K_O- and K_I-inequalities for condensers as well. If $f\colon G \to \mathbf{R}^n$ is discrete open and (A, C) is a condenser in G such that A is a normal domain of f, then (A, C) is called a *normal condenser.* Also the next result is from [V8].

10.15. Theorem. *Suppose that* $f\colon G \to \mathbf{R}^n$ *is a non-constant qr mapping. Then*

$$\text{(1)}\qquad \operatorname{cap}(fA, fC) \le \frac{K_I(f)}{M(f, C)}\operatorname{cap}(A, C)$$

for all condensers (A, C) *in* G *where*

$$M(f, C) = \inf_{y \in fC} \sum_{x \in C \cap f^{-1}(y)} i(x, f)$$

and

$$\text{(2)}\qquad \operatorname{cap}(A, C) \le K_O(f)\, N(f, A)\, \operatorname{cap}(fA, fC)$$

for all normal condensers (A, C) *in* G.

In the next theorem we list some basic properties of quasiregular mappings.

10.16. Theorem. *Let* $f\colon G \to \mathbf{R}^n$ *be a non-constant qr mapping. Then*

(1) $m(B_f) = m(fB_f) = 0$.

(2) $J_f(x) > 0$ a.e. *in* G.

(3) *If* $g\colon G' \to \mathbf{R}^n$ *is a qr mapping with* $fG \subset G'$, *then* $K_O(f \circ g) \leq K_O(f)K_O(g)$ *and* $K_I(f \circ g) \leq K_I(f)K_I(g)$.

10.17. Remarks. Part (3) of 10.16 follows immediately from 10.15. Part (1) of 10.16 can be much improved, see [R10], [MR2], [S2].

In most of our later applications of the K_O- and K_I-inequalities, one may appeal to the following particular cases, which are the transformation formulae for μ_G and λ_G.

10.18. Theorem. *If* $f\colon G \to \mathbf{R}^n$ *is a non-constant qr mapping, then*

(1) $$\mu_{fG}(f(a), f(b)) \leq K_I(f)\, \mu_G(a,b)\ ;\quad a,b \in G\,.$$

In particular, $f\colon (G, \mu_G) \to (fG, \mu_{fG})$ *is Lipschitz continuous. If* $N(f,G) < \infty$, *then*

(2) $$\lambda_G(a,b) \leq K_O(f)\, N(f,G)\, \lambda_{fG}(f(a), f(b))$$

for all $a,b \in G$ *with* $f(a) \neq f(b)$.

Proof. (1) Fix $a,b \in G$ and a curve $\alpha\colon [0,1] \to G$ such that $\alpha(0) = a$,

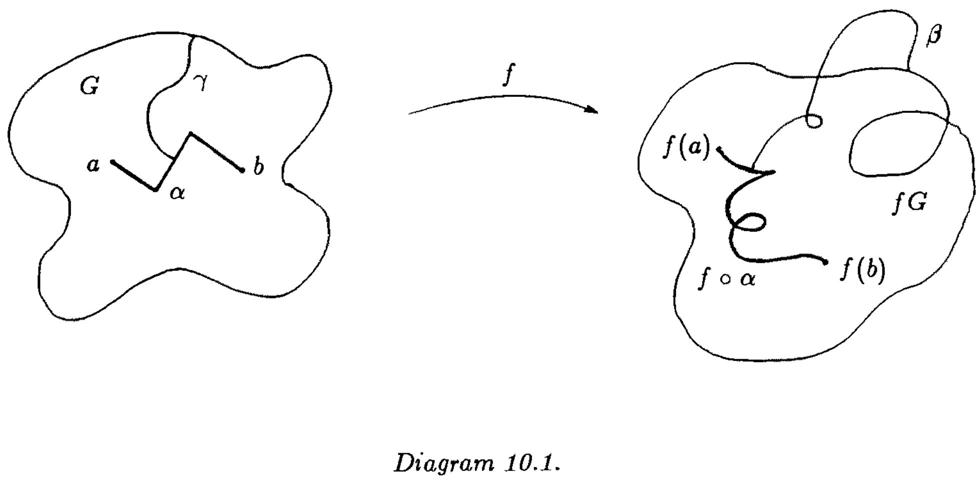


Diagram 10.1.

$\alpha(1) = b$, and denote $\Gamma' = \Delta((f \circ \alpha)[0,1], \partial fG)$. Let Γ be the family of all maximal liftings of the elements of Γ' starting at $|\alpha|$. That is, $\gamma \in \Gamma$ iff there exists β in Γ' such that γ is a maximal lifting of β starting at a point of $|\alpha|$. Then $f\Gamma < \Gamma'$; by the definition (8.3) of the conformal invariant μ_G and by 5.3 and 10.14,

$$\mu_{fG}(f(a), f(b)) \le \mathsf{M}(\Gamma') \le \mathsf{M}(f\Gamma) \le K_I(f)\,\mathsf{M}(\Gamma) .$$

Because $\overline{|\beta|} \cap \partial fG \neq \emptyset$ for all $\beta \in \Gamma'$, it follows from 9.20 that $\overline{|\gamma|} \cap \partial G \neq \emptyset$ for all $\gamma \in \Gamma$. Then by 5.2(2)

$$\mathsf{M}(\Gamma) \le \mathsf{M}(\Delta(|\alpha|, \partial G; G)) .$$

The proof now follows from this and the preceding inequality since α is an arbitrary curve in G with $\alpha(0) = a$, $\alpha(1) = b$ (see (8.4)).

(2) Let $\beta_j\colon [0,1) \to fG$ be paths such that $\beta_j(t) \to \partial fG$, $j = 1,2$, as $t \to 1$, $f(a) = \beta_1(0)$, $f(b) = \beta_2(0)$ and $|\beta_1| \cap |\beta_2| \cap fG = \emptyset$. Let $\gamma_j\colon [0, c_j) \to G$ be a maximal lifting of β_j, $j = 1,2$, with $\gamma_1(0) = a$, $\gamma_2(0) = b$. Since $\beta_j(t) \to \partial fG$ as $t \to 1$ it follows from 9.20 that $\gamma_j(t) \to \partial G$ as $t \to c_j$, $j = 1,2$. Let $\Gamma = \Delta(|\gamma_1|, |\gamma_2|; G)$. By (8.2) and 10.13

$$\lambda_G(a,b) \le \mathsf{M}(\Gamma) \le K_O(f)\,N(f,G)\,\mathsf{M}(f\Gamma) .$$

Because $\beta_j\colon [0,1) \to fG$, $j = 1,2$, were arbitrary curves satisfying the conditions mentioned above and because $f\Gamma \subset \Delta(|\beta_1|, |\beta_2|; G)$, the proof follows from the last inequality, (8.2), and 5.2(2). ❐

10.19. Corollary. *If $f\colon G \to G' = fG$ is a qc mapping, then*

(1) $$\mu_G(a,b)/K_O(f) \le \mu_{fG}(f(a), f(b)) \le K_I(f)\,\mu_G(a,b) ,$$

(2) $$\lambda_G(a,b)/K_O(f) \le \lambda_{fG}(f(a), f(b)) \le K_I(f)\,\lambda_G(a,b)$$

hold for all distinct $a, b \in G$.

Proof. The right inequalities were proved in 10.18. Because $K_O(f^{-1}) = K_I(f)$, $K_I(f^{-1}) = K_O(f)$, the left inequalities also follow from 10.18. ❐

According to 10.18, each qr mapping $f\colon G \to fG$ is a Lipschitz mapping of the (pseudo)metric space (G, μ_G) onto (fG, μ_{fG}). We shall employ the inequalities of Section 8 for $\mathsf{M}(\Delta(E,F))$, which enable us to give a geometric meaning to this general result in many interesting cases and to replace the metric space (G,μ_G) by other metric spaces. Depending on the context, one may wish to replace (G,μ_G) by some less abstract space such as $(\mathbf{B}^n,\rho)$, (G,k_G), (G,j_G) or even $(\mathbf{R}^n,|\,|)$.

If (X,d_X), (Y,d_Y) are (pseudo)metric spaces and $f\colon (X,d_X) \to (Y,d_Y)$ is continuous, then

$$\omega_f(t) = \sup\{\, d_Y(f(x),f(y)) : d_X(x,y) \le t\,\}\,,\ t>0\,, \tag{10.20}$$

is called the *modulus of continuity* of f. This definition clearly depends on the metrics d_X and d_Y. If confusion seems possible we shall specify the metrics. It is clear that $\omega_f\colon (0,\infty) \to (0,\infty]$ is increasing and that $\omega_f(t) \to 0$ as $t \to 0$ iff $f\colon (X,d_X) \to (Y,d_Y)$ is uniformly continuous. A theorem that yields an upper bound for the modulus of continuity is often called a distortion theorem.

One can derive numerous distortion results for qc and qr mappings directly from 10.18, 10.19, and the estimates of Section 8. Examples of such results will be given in Section 11. We shall next give an application of 10.18 which yields a bound for the *linear dilatation* $H(x,f)$ defined by

$$\begin{cases} H(x,f) = \limsup\limits_{r\to 0} \dfrac{L(x,f,r)}{l(x,f,r)}\,, \\ L(x,f,r) = \max\{\,|f(x)-f(z)| : |x-z| = r\,\}\,,\ 0<r<d(x,\partial G)\,, \\ l(x,f,r) = \min\{\,|f(z)-f(x)| : |x-z| = r\,\}\,, \end{cases} \tag{10.21}$$

whenever $f\colon G \to \mathbf{R}^n$ is continuous and $x\in G$.

10.22. Theorem. *If $f\colon G \to \mathbf{R}^n$ is a non-constant qr mapping and $x\in G$, then*

$$H(x,f) \le c\bigl(n,\ K_O(f)\, i(x,f)\bigr) < \infty\,.$$

Proof. We may assume that $x = 0 = f(x)$. Let σ_0 be as in 9.14, $U = U(0,f,\sigma_0)$, and choose $t>0$ such that $\overline{B}^n(3t) \subset U$. For each $r\in(0,t]$ choose $x_r, y_r \in S^{n-1}(r)$ with $|f(x_r)| = L(0,f,r)$, $|f(y_r)| = l(0,f,r)$. Let A_r be the y_r-component of $f^{-1}[0,f(y_r)]$ and B_r the x_r-component of $f^{-1}[f(x_r),\infty)$. Then $0\in A_r$ and $B_r \cap \partial U \ne \emptyset$ by 9.20. Denote $\Gamma_r = \Delta(A_r,B_r;U)$. By 5.9, 5.3, and 5.14 we obtain

$$\mathsf{M}(\Gamma_r) + \omega_{n-1}\Bigl(\log\frac{3t}{r}\Bigr)^{1-n} \ge \mathsf{M}\bigl(\Delta(A_r, U\cap B_r;\mathbf{R}^n)\bigr)\,. \tag{10.23}$$

Next, 7.17 and 5.54(1) yield

$$\mathsf{M}(\Delta(A_r, U\cap B_r; \mathbf{R}^n)) \geq \mathsf{M}(\Delta([0, re_1], [-re_1, -3te_1])) = \tau\Big(\frac{3t+r}{3t-r}\Big) . \tag{10.24}$$

If $|f(x_r)| > |f(y_r)|$ then by 5.27 we obtain

$$\mathsf{M}(f\Gamma_r) \leq \tau\Big(\frac{|f(x_r)|}{|f(y_r)|} - 1\Big) . \tag{10.25}$$

This inequality holds trivially if $|f(x_r)| = |f(y_r)|$. By 10.13, 9.10, and 9.15

$$\mathsf{M}(\Gamma_r) \leq K_O(f)\, i(x,f)\, \mathsf{M}(f\Gamma_r) .$$

We combine the latter inequality with (10.23) and (10.24) and let $r \to 0$. As a result we obtain

$$\tau(1) \leq K_O(f)\, i(0,f)\, \tau(H(0,f) - 1) ,$$

$$H(0,f) \leq 1 + \tau^{-1}\Big(\frac{\tau(1)}{K_O(f)\, i(0,f)}\Big) ,$$

as desired. □

10.26. Corollary. *If* $f\colon \mathbf{R}^n \to \mathbf{R}^n$ *is a qc mapping with* $f(0) = 0$, *then*

$$|f(x)| \leq c(n, K_O(f))|f(y)|$$

for $|x| = |y|$.

Proof. The proof is similar to that of 10.22; in fact, it is slightly simpler. □

10.27. Remark. Making use of the functional identity in 5.53 one can write the constant in 10.22 also as follows

$$c(n,\, K_O(f)\, i(x,f)) = \Big(\gamma^{-1}\Big(\frac{\gamma(\sqrt{2})}{K_O(f)\, i(x,f)}\Big)\Big)^2 .$$

This equality together with 7.51 and 10.22 shows that the linear dilatation $H(x,f)$ of a K-quasiregular mapping f has an upper bound depending only on $K\, i(x,f)$. In particular, this upper bound is independent of n.

10.28. Exercise. Let $f\colon \mathbf{R}^n \to \mathbf{R}^n$ be a K-qc mapping with $f(0) = 0$ and let $m = \min\{\,|f(x)| : |x| = 1\,\}$, $M = \max\{\,|f(x)| : |x| = 1\,\}$. Without appealing to 10.22 or 10.26 show that

$$M/m \leq d(n,K) .$$

[Hint: Let $\Gamma' = \Delta\,(S^{n-1}(m),\, S^{n-1}(M))$. Because $S^{n-1} \cap f^{-1}S^{n-1}(m) \neq \emptyset \neq S^{n-1} \cap f^{-1}S^{n-1}(M)$, 7.34 yields a lower bound for $\mathsf{M}(\Gamma)$, $\Gamma = f^{-1}\Gamma'$.]

10.29. Remark. A. Mori [MOR2] proved that the linear dilatation of a K-qc mapping of a plane domain has an upper bound $e^{\pi K}$. His result was extended to the multidimensional case by F. W. Gehring [G2] who found the upper bound

$$d(n,K)=\exp\Big(2\Big(\frac{K\,\omega_{n-1}}{\gamma(\sqrt{2}\,)}\Big)^{1/(n-1)}\Big)$$

for the linear dilatation of a K-qc mapping of a domain G in $\mathbf{R}^n$. The bound in 10.22 and 10.27 yields a better bound $c(n,K)$ with

$$c(n,K)=\Big[\gamma^{-1}\big(\gamma(\sqrt{2}\,)/K\big)\Big]^2\le d(n,K)/10\,.$$

For these facts see [VU11] and [AVV1]. With a different (larger) constant 10.22 was proved by Yu. G. Reshetnyak [R10] and O. Martio, S. Rickman, and J. Väisälä [MRV1].

10.30. Exercise. Show that $d(2,K)=e^{\pi K}$. Next using 7.26(1) show that $c(2,K)<d(2,K)/10$. Applying 7.47 and 7.50 find a dimension-independent upper bound for $c(n,K)$.

10.31. Remark. The sharp upper bound $\lambda(K)$ for the linear dilatation of a K-qc mapping of $\mathbf{R}^2$ onto $\mathbf{R}^2$ was found by O. Lehto, K. I. Virtanen, and J. Väisälä [LVV]. For further results of this type see [HEL]. It can be shown that

$$\lambda(K)=c(2,K)-1=\frac{\varphi^2_{K,2}(1/\sqrt{2}\,)}{\varphi^2_{1/K,2}(1/\sqrt{2}\,)}$$

(see [LVV] and 5.61(2)) and that $\lambda(K)\sim\frac{1}{16}e^{\pi K}$ for large values of K. It can be shown (cf. [AVV3]) that

$$e^{\pi(K-1)}\le\lambda(K)\le e^{\pi(K-1/K)}$$

for $K\ge1$.

10.32. Notes. A thorough study of the K_O- and K_I-inequalities is contained in [RI12]. Theorem 10.18 and Corollary 10.19 are from [VU10], Theorem 10.22 from [VU11].

10.33. Remark. Quasiregular mappings have important normal family properties, which were established by Yu. G. Reshetnyak [R5] (for a simple proof see P. Lindqvist [LI1]). These properties will not be used in this book.

11. Distortion theory

In the present section we shall put into effective use the transformation formulae 10.18(1) and (2) for the conformal invariants λ_G and μ_G. Most results of this section are of the following general type: we combine the transformation formulae in 10.18 with some particular estimates for λ_G and μ_G proved in Chapter II and as a result obtain distortion theorems. Besides the fundamental distortion theorems, the qr variant of the Schwarz lemma, and the Hölder continuity, we prove several additional special distortion theorems.

11.1. Theorem. *Let* $E \subset \overline{\mathbf{R}}^n$ *be a compact set of positive capacity and let* $f\colon \mathbf{B}^n \to \overline{\mathbf{R}}^n \setminus E$ *be a* K*-qm mapping. Then*

$$q(f(x), f(y)) \le \frac{a\,K}{c(E)}\,\mu_{\mathbf{B}^n}(x,y) \le \frac{b\,K}{c(E)}\left(-\log \operatorname{th} \tfrac{1}{4}\rho(x,y)\right)^{1-n}$$

for distinct $x, y \in \mathbf{B}^n$ *where* a *and* b *depend only on* n.

Proof. It follows from 6.1 and 8.5 that

$$\begin{aligned}\mu_{f\mathbf{B}^n}(f(x), f(y)) &\ge d_4 \min\{\, c(E)\,,\, q(f(x), f(y))\,\} \\ &\ge d_4\, q(f(x), f(y)) \min\{\, d_3\,,\, c(E)\,\}\,.\end{aligned}$$

Because E is of positive capacity we deduce from 6.1 that $1 \ge c(E)/c(\overline{\mathbf{R}}^n) \ge c(E)/d_2 > 0$, and therefore

$$\mu_{f\mathbf{B}^n}(f(x), f(y)) \ge d_4\, c(E)\, q(f(x), f(y)) \min\{\, d_3/d_2\,,\, 1\,\}\,.$$

The proof follows now from 10.18(1), 8.6(1), and (7.30). □

It follows from 11.1 and the monotone property 6.1(2) of the set function $c(E)$ that for fixed K and $\mu_{\mathbf{B}^n}(x,y)$, the distance $q(f(x), f(y))$ decreases if the set E becomes larger. In other words, the larger the set omitted by the mapping f, the less f can oscillate as a mapping between metric spaces $f\colon (\mathbf{B}^n, \mu_{\mathbf{B}^n}) \to (\overline{\mathbf{R}}^n, q)$. Later on we shall encounter a similar phenomenon with other metric spaces in place of $(\mathbf{B}^n, \mu_{\mathbf{B}^n})$ and $(\overline{\mathbf{R}}^n, q)$.

The next result is a counterpart of the Schwarz lemma for qr mappings. We consider here the function $\varphi_K = \varphi_{K,n}$ introduced in (7.44).

11.2. Theorem. *Let* $f\colon \mathbf{B}^n \to \mathbf{R}^n$ *be a non-constant* K*-qr mapping with* $f\mathbf{B}^n \subset \mathbf{B}^n$ *and let* $\alpha = K_I(f)^{1/(1-n)}$ *. Then*

$$\text{th}\,\tfrac{1}{2}\rho(f(x),f(y)) \le \varphi_K\left(\text{th}\,\tfrac{1}{2}\rho(x,y)\right) \le \lambda_n^{1-\alpha}\left(\text{th}\,\tfrac{1}{2}\rho(x,y)\right)^{\alpha} \,, \tag{1}$$

$$\rho(f(x),f(y)) \le K_I(f)\left(\rho(x,y)+\log 4\right) \,, \tag{2}$$

hold for all $x,y \in \mathbf{B}^n$ *, where* λ_n *is the constant in* (7.21).

Proof. Fix $x,y\in\mathbf{B}^n$. Because $f\mathbf{B}^n \subset \mathbf{B}^n$ it follows from 8.5, 8.6, and (7.32) that

$$\mu_{f\mathbf{B}^n}(f(x),f(y)) \ge \mu_{\mathbf{B}^n}(f(x),f(y)) = \gamma(1/\,\text{th}\,b)$$

where $b = \frac{1}{2}\rho(f(x),f(y))$. Similarly, by 10.18(1) and 8.6,

$$\mu_{f\mathbf{B}^n}(f(x),f(y)) \le K_I(f)\,\mu_{\mathbf{B}^n}(x,y) = K_I(f)\,\gamma(1/\,\text{th}\,a)$$

where $a = \frac{1}{2}\rho(x,y)$. These inequalities together with 7.47(1) imply (1). For the proof of (2) we note that by (7.31) and 10.18(1)

$$A\,\rho(f(x),f(y)) \le \gamma(1/\,\text{th}\,b) \le K_I(f)\,A\left(\rho(x,y)+\log 4\right)$$

where $A = 2^{n-1}c_n$. Hence we have proved also (2). ❐

11.3. Corollary. *Let* $f\colon \mathbf{B}^n \to \mathbf{B}^n$ *be a* K*-qr mapping with* $f(0)=0$ *and let* $\alpha = K_I(f)^{1/(1-n)}$ *. Then*

$$|f(x)| \le \varphi_{K,n}(|x|) \le \lambda_n^{1-\alpha}|x|^{\alpha} \le 2^{1-1/K}K|x|^{1/K} \,, \tag{1}$$

$$|f(x)| \le \frac{a-1}{a+1}\,, \quad a = \left(4\,\frac{1+|x|}{1-|x|}\right)^{K_I(f)} \,, \tag{2}$$

for all $x \in \mathbf{B}^n$.

Proof. Apply (2.17) and 11.2 with $y=0$ and recall that $\lambda_n^{1-\alpha} \le 2^{1-1/K}K$ by 7.51. ❐

The following invariance properties of 11.1 and 11.2 should be noted. The inequality of 11.1 yields the same upper bounds for

$$q(f(x),f(y)) \quad\text{and}\quad q((h\circ f\circ g_1)(x),\,(h\circ f\circ g_1)(y)) \,,$$

while the second one yields the same upper bounds for

$$\rho(f(x), f(y)) \quad \text{and} \quad \rho((g_1 \circ f \circ g_2)(x), (g_1 \circ f \circ g_2)(x))$$

whenever $g_1, g_2 \in \mathcal{M}(\mathbf{B}^n)$ and h is a sense-preserving spherical isometry.

It should also be observed that the explicit estimate 11.3(1) is sharp if $K = 1$.

In 11.2 we assumed that $f\mathbf{B}^n \subset \mathbf{B}^n$ and proved that $f\colon (\mathbf{B}^n, \rho) \to (\mathbf{B}^n, \rho)$ is uniformly continuous with a quantitative bound for its modulus of continuity. If, in addition, $\mathbf{B}^n \setminus f\mathbf{B}^n \neq \emptyset$, one would expect a better result than 11.2. For instance, one could hope to replace the target space $(\mathbf{B}^n, \rho)$ in 11.2 by $(f\mathbf{B}^n, k_{f\mathbf{B}^n})$. In the particular case of Möbius transformations this indeed is possible by 3.9 (later on we shall prove that this is possible also for quasiconformal mappings). Now we are going to show that for qr mappings and even for bounded analytic functions such an expectation is futile.

11.4. Example. Let $g\colon \mathbf{B}^2 \to \mathbf{B}^2 \setminus \{0\} = g\mathbf{B}^2$ be the exponential function $g(z) = \exp(\frac{z+1}{z-1})$, $z \in \mathbf{B}^2$. We shall show that $g\colon (\mathbf{B}^2, \rho) \to (g\mathbf{B}^2, k_{g\mathbf{B}^2})$ fails to be uniformly continuous. To this end, let $x_j = (e^j - 1)/(e^j + 1)$, $j = 1, 2, \ldots$. It follows from (2.17) that $\rho(0, x_j) = j$ and thus $\rho(x_j, x_{j+1}) = 1$. Let $Y = \mathbf{B}^2 \setminus \{0\}$. Since $g(x_j) = \exp(-e^j)$ we get by (3.4) and (2.34)

$$\begin{aligned} k_Y(g(x_j), g(x_{j+1})) &\geq j_Y(g(x_j), g(x_{j+1})) \\ &= \log\left[1 + \left(\exp e^{j+1}\right)\left(\exp(-e^j) - \exp(-e^{j+1})\right)\right] \\ &= \log\left[1 + \exp(e^{j+1} - e^j) - 1\right] = e^{j+1} - e^j \to \infty \end{aligned}$$

as $j \to \infty$. In conclusion, $g\colon (\mathbf{B}^2, \rho) \to (Y, k_Y)$ cannot be uniformly continuous, because $\rho(x_j, x_{j+1}) = 1$.

In this example $\partial(g\mathbf{B}^2)$ consists of a point component $\{0\}$ and the unit circle $\partial\mathbf{B}^2$. We now show that if each boundary component of the image domain is non-degenerate, then the situation will be different, at least under an additional condition. Later on we shall show that this additional condition, which requires that the image domain be uniform, can in fact be removed, and that the exponential function in 11.3 is in a sense an extremal case.

11.5. Theorem. *Let $f\colon \mathbf{B}^n \to \mathbf{R}^n$ be a non-constant qr mapping, let $E \subset \overline{\mathbf{R}}^n \setminus f\mathbf{B}^n$ be a non-degenerate continuum such that $\infty \in E$, and let $G = \mathbf{R}^n \setminus E$ be a domain.*

(1) *Then* $f\colon (\mathbf{B}^n, \rho) \to (G, j_G)$ *is uniformly continuous.*

(2) *If* G *is uniform, then* $f\colon (\mathbf{B}^n, \rho) \to (G, k_G)$ *is uniformly continuous.*

Proof. (1) The proof follows the same general pattern as the one in 11.2. The particular estimates needed for the present case are supplied by 7.41, 8.6, and (7.30).

(2) The proof follows from (1) and the definition 3.8 of a uniform domain. ❐

11.6. Lemma. *Let* G *and* G' *be proper subdomains of* $\mathbf{R}^n$ *, where* G *is uniform and* G' *has connected complement* $\overline{\mathbf{R}}^n \setminus G'$ *. If* $f\colon G \to \mathbf{R}^n$ *is a qr mapping with* $fG \subset G'$ *, then for all* $x, y \in G$

$$j_{G'}\big(f(x), f(y)\big) \le a_1\, j_G(x,y) + a_2 \ ,$$

where a_1, a_2 *are positive numbers depending only on* n *,* $K_I(f)$ *, and the constant in the definition of a uniform domain.*

Proof. By 8.31, 10.18(1), 8.30(2), and 3.8 we obtain

$$\begin{aligned} c_n\, j_{G'}\big(f(x), f(y)\big) &\le \mu_{G'}\big(f(x), f(y)\big) \le K_I(f)\, \mu_G(x,y) \\ &\le K_I(f)\big(b_1\, k_G(x,y) + b_2\big) \\ &\le K_I(f)\, b_1 A\, j_G(x,y) + K_I(f)\, b_2 \ . \quad ❐ \end{aligned}$$

11.7. Exercise. Show that the hypothesis that $\mathbf{R}^n \setminus G'$ be connected cannot be removed from 11.6 if $n = 2$ and $G = \mathbf{B}^n$. [Hint: Show that the exponential function in 11.4 provides a counterexample in the present case, too. Recall that $\rho \sim j_{\mathbf{B}^n}$ by 2.41(1).]

11.8. Exercise. Observe first that 11.2(1) and (2) hold also for a qr mapping $f\colon \mathbf{B}^n \to \mathbf{H}^n$. Show that

$$|f(x)| \le 2^{2\beta} |f(0)| \Big(\frac{1+|x|}{1-|x|}\Big)^{\beta} \ , \quad \beta = K_I(f) \ ,$$

for a qr mapping $f\colon \mathbf{B}^n \to \mathbf{H}^n$ when $x \in \mathbf{B}^n$. [Hint: Apply 11.2(2) for a qr mapping of $\mathbf{B}^n$ into $\mathbf{H}^n$ and the inequality $\rho_{\mathbf{H}^n}(x,y) \ge |\log(|x|/|y|)|$, $x, y \in \mathbf{H}^n$. The required inequality then follows from 2.41(2) and (2.39).]

11.9. Exercise. Show that if $f\colon \mathbf{B}^n \to \mathbf{B}^n$ is K-qr, then for all $x \in \mathbf{B}^n$

$$1 - |f(x)| \geq 2^{-2K}(1 - |f(0)|)\Big(\frac{1-|x|}{1+|x|}\Big)^K .$$

[Hint: Observe that by 2.36(1) and 2.41(1)

$$\rho(x,y) \geq j_{\mathbf{B}^n}(x,y) \geq \log\frac{1-|y|}{1-|x|}$$

for all $x, y \in \mathbf{B}^n$. Now apply 11.2(2) and (2.17).]

11.10. Theorem. *Suppose that $f\colon G \to \mathbf{R}^n$ is a bounded qr mapping and that F is a compact subset of G. Let $\alpha = K_I(f)^{1/(1-n)}$ and $C = \lambda_n^{1-\alpha} d(fG)/d(F,\partial G)^\alpha$ where λ_n is as in (7.21). Then f satisfies the Hölder condition*

$$|f(x) - f(y)| \leq C\,|x-y|^\alpha \tag{11.11}$$

for $x \in F$, $y \in G$.

Proof. Set $r = d(F,\partial G)$. Suppose first that $|x-y| < r$. Define $g\colon \mathbf{B}^n \to \mathbf{B}^n$ by

$$g(z) = \frac{f(x+rz) - f(x)}{d(fG)} .$$

Then $g(0) = 0$ and $K_I(g) = K_I(f)$ by 10.16(3). By 11.2 we get $|g(z)| \leq \lambda_n^{1-\alpha}|z|^\alpha$. Setting $z = (y-x)/r$ we obtain (11.11). Next assume that $|x-y| \geq r$. Since $\lambda_n \geq 1$ (in fact, $\lambda_n \geq 4$, see 7.22) we have

$$|f(x) - f(y)| \leq d(fG) \leq r^{-\alpha} d(fG)\,|x-y|^\alpha \leq C|x-y|^\alpha . \quad \square$$

11.12. Theorem. *Let $f\colon \mathbf{B}^n \to \mathbf{B}^n$ be a K-qr mapping into $\mathbf{B}^n$. Then*

$$|f(x) - f(y)| \leq b_K\left(\operatorname{th}\tfrac{1}{2}\rho(x,y)\right)$$

for all $x, y \in \mathbf{B}^n$, where $b_K(s) = 2\varphi_{K,n}(s)/\big(1+\sqrt{1-\varphi_{K,n}^2(s)}\,\big)$. The result is sharp if f is a rotation fixing the origin and $x = -y$.

Proof. If we let $t' = \frac{1}{2}\rho\big(f(x), f(y)\big)$, it follows from (2.27) and 2.29(2) that

$$|f(x) - f(y)| \leq 2\operatorname{th}\tfrac{1}{2}t' = \frac{2\operatorname{th} t'}{1+\sqrt{1-\operatorname{th}^2 t'}} .$$

The desired inequality follows now from 11.2(1). Since $\varphi_{1,n}(r) = r$, the sharpness assertion follows from the one in (2.27). $\square$

11.13. Corollary. *Under the assumptions of* 11.12

$$|f(x) - f(y)| \le \varphi_{K,n}(a) + \varphi_{K,n}^2(a)$$

where $a = \operatorname{th} \frac{1}{2}\rho(x,y)$ *for all* $x, y \in \mathbf{B}^n$.

Proof. The proof follows from 11.12 and the inequality

$$\frac{2}{1+\sqrt{1-x^2}} \le 1 + x\,, \quad 0 \le x \le 1\,. \quad \square$$

11.14. Theorem. *For* $n \ge 2$, $r \in (0,1)$, *and* $K \in [1,\infty)$ *there exists a number* $a(r)$ *with* $\lim_{r\to 0} a(r) = 1$ *such that if* $f\colon \mathbf{B}^n \to \mathbf{B}^n$ *is a* K-*qr mapping into* $\mathbf{B}^n$, *then*

$$|f(x) - f(y)| \le a(r)\, \lambda_n^{1-\alpha} |x-y|^{\alpha}$$

for all $x, y \in \overline{B}^n(r)$ *where* $\alpha = K^{1/(1-n)}$.

Proof. Let $r \in (0,1)$ and $x, y \in \overline{B}^n(r)$. Then

$$\operatorname{th} \tfrac{1}{2}\rho(x,y) \le \operatorname{th} \tfrac{1}{2}\rho(-re_1, re_1) = \frac{2r}{1+r^2}$$

by 2.47. By the inequality in the proof of 11.13, by 11.12, 11.2(1), and 2.47 we obtain

$$\begin{aligned} |f(x) - f(y)| \le b_K\left(\operatorname{th} \tfrac{1}{2}\rho(x,y)\right) &\le \frac{2\,\lambda_n^{1-\alpha}(\operatorname{th} \frac{1}{2}\rho(x,y))^{\alpha}}{1+\sqrt{1-\varphi_{K,n}^2(2r/(1+r^2))}} \\ &\le \frac{\lambda_n^{1-\alpha}\,[1+\varphi_{K,n}(2r/(1+r^2))]\,|x-y|^{\alpha}}{\left[|x-y|^2 + (1-|x|^2)(1-|y|^2)\right]^{\alpha/2}}\,. \end{aligned}$$

We may choose

$$a(r) = \Big(1 + \varphi_{K,n}\Big(\frac{2r}{1+r^2}\Big)\Big)(1-r^2)^{-\alpha}\,. \quad \square$$

The following result is a generalization of Liouville's theorem concerning the growth of entire analytic functions.

11.15. Theorem. *Suppose that* $f\colon \mathbf{R}^n \to \mathbf{R}^n$ *is a qr mapping and that* $\lim_{x\to\infty} |x|^{-\alpha}|f(x)| = 0$ *where* $\alpha = K_I(f)^{1/(1-n)}$. *Then* f *is a constant.*

Proof. We can write $|f(x)| \le |x|^\alpha \epsilon(|x|)$ where $\epsilon(R) \to 0$ as $R \to \infty$. Fix $x \in \mathbf{R}^n$ and choose $R > |x|$. Applying (11.11) for $G = B^n(R)$ and $F = \{0\}$ we obtain $|f(x) - f(0)| \le C|x|^\alpha$ where $C = \lambda_n^{1-\alpha} d(F, \partial G)^{-\alpha} d(fG) \le 2\lambda_n^{1-\alpha}\epsilon(R)$. Thus $|f(x) - f(0)| \le 2\lambda_n^{1-\alpha}\epsilon(R)|x|^\alpha$. Letting $R \to \infty$ yields $f(x) = f(0)$. Hence f is a constant. ❒

11.16. Remarks. The exponent α in 11.2(1) and 11.15 is best possible. As to 11.2(1), the function $f : \mathbf{B}^n \to \mathbf{B}^n$, $f(x) = x|x|^{\alpha-1}$, $x \in \mathbf{B}^n$, $K_I(f) = K(f) = \alpha^{1-n}$, is a desired example (see [V7, 16.2] for the calculation of $K_I(f)$). The same function, as a mapping of $\mathbf{R}^n$ onto $\mathbf{R}^n$, shows that the condition $\lim_{x\to\infty} |x|^{-\alpha}|f(x)| = 0$ in 11.15 cannot be replaced by the requirement that $|x|^{-\alpha}|f(x)|$ be bounded.

For $\varphi \in (0, \frac{1}{2}\pi)$ let $C(\varphi) = \{z \in \mathbf{R}^n : z \cdot e_n \le |z| \cos\varphi\}$. We next give a formulation of 11.2 for maps into a cone or into an infinite cylinder.

11.17. Theorem. *Let $f\colon \mathbf{B}^n \to \mathbf{R}^n$ be a non-constant qr mapping.*

(1) *If $\varphi \in (0, \frac{1}{2}\pi)$ and $f\mathbf{B}^n \subset C(\varphi)$, then for all $x \in \mathbf{B}^n$*

$$|f(x)| \le |f(0)|\, 4^{a\varphi} \Big(\frac{1+|x|}{1-|x|}\Big)^{a\varphi}$$

where a depends only on n and $K_I(f)$.

(2) *If $f\mathbf{B}^n \subset \{ x \in \mathbf{R}^n : x_1^2 + \ldots + x_{n-1}^2 < 1 \}$, then for all $x, y \in \mathbf{B}^n$*

$$|f(x)| \le |f(y)| + A\, K_I(f)\, (\rho(x,y) + \log 4)$$

where A is a positive constant depending only on n.

Proof. (1) By 5.29, 10.18(1), 8.6(1), (7.31), and 7.26(2) we obtain

$$\begin{aligned} \frac{d_n}{\varphi} \log \frac{|f(x)|}{|f(0)|} &\le \mu_{f\mathbf{B}^n}(f(x), f(0)) \le K_I(f)\, \mu_{\mathbf{B}^n}(x, 0) \\ &\le K_I(f)\, 2^{n-1} c_n \log\Big(4\, \frac{1+|x|}{1-|x|}\Big)\,. \end{aligned}$$

The proof of (1) with $a = 2^{n-1} c_n K_I(f)/d_n$ follows.

(2) Assume first that $|f(x)| > |f(y)| + 1$. From 5.29 we deduce that

$$\mu_{f\mathbf{B}^n}(f(x), f(y)) \ge d_n \int\limits_{|f(y)|+1}^{|f(x)|} \frac{dr}{\varphi(r) r} \ge \frac{2d_n}{\pi}(|f(x)| - |f(y)| - 1)\,.$$

Here $\varphi(r) \in (0, \frac{1}{2}\pi)$ is such that

$$S^{n-1}(r) \cap \{ x \in \mathbf{R}^n \,:\, x_1^2 + \ldots + x_{n-1}^2 < 1 \} = S^{n-1}(r) \cap C(\varphi(r))$$

for $r > 1$, i.e. $\varphi(r) = \arcsin(1/r)$ and $r\varphi(r) < \frac{1}{2}\pi$. By 10.18(1) and (7.31) we obtain as in the proof of (1)

$$\begin{aligned} |f(x)| &\le |f(y)| + 1 + TK_I(f)\big(\rho(x,y) + \log 4\big) \\ &\le |f(y)| + A\,K_I(f)\big(\rho(x,y) + \log 4\big) \end{aligned}$$

where $T = 2^{n-2}c_n\pi/d_n$ and $A = T + 1/\log 4$. Since equality holds for $|f(x)| \le |f(y)| + 1$ as well, the proof of (2) is complete. ❐

11.18. Remark. For small values of $\rho(x,y)$ one can improve 11.17 by applying 7.26(1) instead of 7.26(2). Recall also 7.28(1).

11.19. Theorem. *Let $f\colon \mathbf{B}^n \to \mathbf{B}^n$ be a qr mapping with $N(f,\mathbf{B}^n) = N < \infty$. Then*

$$\operatorname{th} \tfrac{1}{4}\rho\big(f(x), f(y)\big) \le 2\left(\operatorname{th} \tfrac{1}{4}\rho(x,y)\right)^{\beta}$$

holds for all $x, y \in \mathbf{B}^n$ where $\beta = 1/(NK_O(f))$. Furthermore, if $f(0) = 0$, then for all $x \in \mathbf{B}^n$

$$\frac{|f(x)|}{1 + \sqrt{1 - |f(x)|^2}} \le 2\Big(\frac{|x|}{1 + \sqrt{1 - |x|^2}}\Big)^{\beta}.$$

Proof. We may assume that $f(x) \ne f(y)$. It follows from 8.6(2) and 8.7 that

$$\lambda_{\mathbf{B}^n}(x,y) = \tfrac{1}{2}\tau\big(\operatorname{sh}^2 \tfrac{1}{2}\rho(x,y)\big) \ge -c_n \log \operatorname{th} \tfrac{1}{4}\rho(x,y)\,. \tag{11.20}$$

Because $f\mathbf{B}^n \subset \mathbf{B}^n$, it follows from 8.5, 8.6(2), and 8.7 that

$$\lambda_{f\mathbf{B}^n}\big(f(x), f(y)\big) \le \lambda_{\mathbf{B}^n}\big(f(x), f(y)\big) < c_n \log \frac{2}{\operatorname{th} \frac{1}{4}\rho(f(x), f(y))}\,. \tag{11.21}$$

The proof now follows from (11.20), (11.21), and 10.18(2). If $f(0) = 0$, the assertion follows from the above inequality and (2.17), 2.29(2). ❐

11.22. Exercise. Observe first that the proof of 11.19 yields the inequality

$$\operatorname{sh}^2 b \le \tau^{-1}\Big(\frac{\tau(\operatorname{sh}^2 a)}{NK_O(f)}\Big)$$

where $a = \frac{1}{2}\rho(x,y)$ and $b = \frac{1}{2}\rho\big(f(x), f(y)\big)$. Next assume, in addition, that $f(0) = 0$ and $N = 1$. Exploiting the functional identity 5.53 and the definition (7.45) show that the above inequality with $y = 0$ yields

$$|f(x)|^2 \le 1 - \varphi_{1/K,n}^2\big(\sqrt{1 - |x|^2}\,\big)$$

for all $x \in \mathbf{B}^n$. (Compare this to the Schwarz lemma 11.3.)

11.23. Exercise. Assume that $f\colon \mathbf{B}^n \to \mathbf{B}^n$ is K–qc with $f(0) = 0$ and $f\mathbf{B}^n = \mathbf{B}^n$. Show that

$$|f(x)|^2 \le \min\{\, \varphi_{K,n}^2(|x|)\,,\, 1 - \varphi_{1/K,n}^2\big(\sqrt{1-|x|^2}\,\big)\,\}\,,$$
$$|f(x)|^2 \ge \max\{\, \varphi_{1/K,n}^2(|x|)\,,\, 1 - \varphi_{K,n}^2\big(\sqrt{1-|x|^2}\,\big)\,\}\,.$$

[Hint: Apply 11.22 and 11.3 also to f^{-1}.] Recall that in the case $n = 2$ we have $\varphi_{K,2}^2(r) = 1 - \varphi_{1/K,2}^2(\sqrt{1-r^2})$ for all $K > 0$ and $0 < r < 1$ by 5.61(2) while the analogous relation fails to hold for $n \ge 3$ by 7.58.

11.24. Theorem. *Let* $f\colon \mathbf{B}^n \to \mathbf{R}^n \setminus \{0\}$ *be a qr mapping with* $N(f, \mathbf{B}^n) \le p < \infty$. *Then for* $x, y \in \mathbf{B}^n$

$$|f(x)| \le |f(y)|\,\big(1 + \tau^{-1}\,\big(A\,\tau(\operatorname{sh}^2 \tfrac{1}{2}\rho(x,y))\big)\big)\,,$$

where $A = 1/(2pK_O(f))$.

Proof. If $|f(x)| \le |f(y)|$ there is nothing to prove. Hence we may assume that $|f(x)| > |f(y)|$. By 5.27 and 8.5 we obtain

$$\begin{aligned}\lambda_{f\mathbf{B}^n}\big(f(x), f(y)\big) &\le \lambda_G\big(f(x), f(y)\big) \le \mathsf{M}\big(\Delta([0, f(y)], [f(x), \infty))\big)\\ &\le \tau\Big(\frac{|f(x)|}{|f(y)|} - 1\Big)\end{aligned}$$

where $G = \mathbf{R}^n \setminus \{0\}$. Next, by 8.6(2)

$$\lambda_{\mathbf{B}^n}(x,y) = \tfrac{1}{2}\,\tau\big(\operatorname{sh}^2 \tfrac{1}{2}\rho(x,y)\big)$$

and by 10.18(2)

$$\lambda_{\mathbf{B}^n}(x,y) \le p\,K_O(f)\,\lambda_{f\mathbf{B}^n}\big(f(x), f(y)\big)\,.$$

The desired bound follows from these relations. □

We require the following important theorem of Martio, Rickman, and Väisälä [MRV3, 2.3] on locally homeomorphic qr maps of $\mathbf{B}^n$, $n \ge 3$. The proof of this theorem makes use of an ingenious method of V. A. Zorich [ZO1]. The proof will be omitted. A similar result for qm mappings was proved by Martio and Srebro [MSR4].

11.25. Theorem. *For* $n \ge 3$ *and* $K \ge 1$ *there exists a number* $\psi = \psi(n, K) \in (0,1)$ *such that every locally homeomorphic* K*–qr mapping* $f\colon \mathbf{B}^n \to \mathbf{R}^n$ *is injective in* $B^n(x,\, (1-|x|)\psi)$ *for all* $x \in \mathbf{B}^n$.

11.26. Exercise. Applying (2.23) show that $D(x,M) \subset B^n(x, T(1-|x|))$ where $T = (2\,\mathrm{th}\,\frac{1}{2}M)/(1-\mathrm{th}\,\frac{1}{2}M)$, $|x| < 1$. Conversely show that $D(z,M) \subset B^n(z, (1-|z|)\psi)$ where $|z| < 1$, $\psi \in (0,1)$, $M = 2\,\mathrm{arth}(\psi/(2+\psi))$.

11.27. Theorem. *Let $f\colon \mathbf{B}^n \to \mathbf{R}^n\setminus\{0\}$ be a locally homeomorphic qr mapping and $n \ge 3$. Then*

$$|f(x)| \le C|f(0)|\Big(\frac{1+|x|}{1-|x|}\Big)^a$$

where C and a are positive numbers depending only on n and $K(f)$.

Proof. Let $\psi = \psi(n, K(f))$ be as in 11.25 and define $g_x(z) = f_x(x+z(1-|x|)\psi)$ for $z \in \mathbf{B}^n$ and $x \in \mathbf{B}^n$. Then g_x is injective and K–qc in $\mathbf{B}^n$ by 11.25.

We are going to show first that $|f(x)|$ satisfies the Harnack inequality (4.11) in $\mathbf{B}^n$ with $s \in (0, \frac{1}{2}\psi]$ and

$$C_s = 1 + \tau^{-1}(A\,\tau(16/9)) \,, \quad A = 1/(2K_O(f)) \,. \tag{11.28}$$

To this end let $B^n(z,r) \subset \mathbf{B}^n$ and $x_1, x_2 \in \overline{B}^n(z, sr)$, $s \in (0, \frac{1}{2}\psi]$. By 11.24 we obtain

$$\frac{|f(x_1)|}{|f(x_2)|} = \frac{|g_z(y_1)|}{|g_z(y_2)|} \le 1 + \tau^{-1}\left(A\,\tau(\mathrm{sh}^2\,\tfrac{1}{2}\rho(y_1,y_2))\right)$$

where $y_j = (x_j - z)/((1-|z|)\psi) \in \mathbf{B}^n$ and $A = 1/(2K_O(f))$. Because $|y_j| \le \frac{1}{2}$ for $j = 1,2$ it follows from (2.17) that

$$2B = \rho(y_1, y_2) \le 2\log 3 \,,$$

and hence $\mathrm{sh}^2\, B \le 16/9$. We have thereby proved (11.28).

By virtue of (11.28) and 4.12 we obtain

$$|f(x)| \le C_s^{1+t}|f(0)|$$

where C_s is as in (11.28) and $t = (\log\frac{1+|x|}{1-|x|})/\log\frac{1+s}{1-s}$, $s = \frac{1}{2}\psi$. We have thus proved the desired inequality with $C = C_s$ and $a = (\log C_s)/\log\frac{2+\psi}{2-\psi}$. (Recall the relationship (10.4) between $K_O(f)$ and $K(f)$.) ☐

11.29. Exercise. A counterexample to show that 11.25 is false for $n = 2$ is easily found. Denote $f_j(z) = \exp(jz)$, $j = 2,3,\ldots$, $z \in \mathbf{B}^2$. By considering the family $\{f_j\}$ we see that 11.25 is false for $n = 2$. Find a counterexample to show that 11.27, too, fails to hold for $n = 2$.

Next we shall study the behavior of the function

$$r_G(x,y) = \frac{|x-y|}{\min\{d(x),\, d(y)\}}$$

under quasiconformal mappings.

11.30. Theorem. *Let G and G' be proper subdomains of $\mathbf{R}^n$ and $f\colon \mathbf{R}^n \to \mathbf{R}^n$ a K–qc homeomorphism such that $fG = G'$. Then for all $x,y \in G$*

$$r_{G'}\big(f(x),f(y)\big) \le \tau^{-1}\Big(\frac{1}{4K}\,\tau\big(r_G(x,y)\big)\Big)\,.$$

Proof. We may assume that $d(f(x),\partial G') \le d(f(y),\partial G')$. Fix $z' \in \partial G'$ such that $|f(x)-z'| = d(f(x),\partial G')$ and $z \in \partial G$ such that $f(z) = z'$. Then by 10.18(2)

$$\lambda_D(x,y) \le K\,\lambda_{D'}\big(f(x),f(y)\big)\,,$$

where $D = \mathbf{R}^n \setminus \{z\}$ and $D' = fD = \mathbf{R}^n \setminus \{z'\}$. By 8.24

$$\lambda_D(x,y) \ge \tau\Big(\frac{|x-y|}{|x-z|}\Big) \ge \tau\big(r_G(x,y)\big)\,,$$

$$\lambda_{D'}\big(f(x),f(y)\big) \le 4\,\tau\Big(\frac{|f(x)-f(y)|}{|f(x)-z'|}\Big) = 4\,\tau\big(r_{G'}(f(x),f(y))\big)\,.$$

The desired result follows immediately from the above inequalities. ❐

Applying Theorem 11.30 with $G = \mathbf{R}^n \setminus \{0\}$ yields the following result.

11.31. Corollary. *Let $f\colon \mathbf{R}^n \to \mathbf{R}^n$ be a K–qc mapping with $f(0) = 0$. Then for $x,y \in \mathbf{R}^n \setminus \{0\}$*

$$\frac{|f(x)-f(y)|}{\min\{|f(x)|,\,|f(y)|\}} \le \tau^{-1}\left(\frac{1}{4K}\,\tau\Big(\frac{|x-y|}{\min\{|x|,|y|\}}\Big)\right)\,.$$

Next we shall prove a result where f need not be defined on the whole space $\mathbf{R}^n$ as it was in 11.30 and 11.31.

11.32. Theorem. *Let G be a proper subdomain of $\mathbf{R}^n$, suppose that G is c-QED, and let $f\colon G \to fG$ be K–qc with $fG \subset \mathbf{R}^n$. Then*

$$r_{fG}\big(f(x),f(y)\big) \le \tau^{-1}\Big(\frac{c}{2^{n+1}K}\,\tau\big(r_G(x,y)\big)\Big)$$

for all $x,y \in G$.

Proof. By 8.29 we obtain

$$\lambda_G(x,y) \geq c\,\tau(r^2+2r) \geq 2^{1-n} c\,\tau(r)$$

where $r = r_G(x,y)$. Next by 8.25 we get

$$\lambda_{fG}(f(x),f(y)) \leq 4\,\tau\big(r_{fG}(f(x),f(y))\big)\,.$$

The desired inequality now follows easily. □

11.33. Example. We shall now show that the c-QED condition in 11.32 is necessary. Let $G = \mathbf{B}^2 \setminus [0,e_1)$ and let $f\colon G \to \mathbf{B}^2_+ = \mathbf{B}^2 \cap \mathbf{H}^2$ be the conformal map $f(z) = \sqrt{z}$, $z \in G$. Let $x_j = (1/2, 1/j)$, $y_j = (1/2, -1/j)$, $j = 4,5,\ldots$. Then $r_G(x_j,y_j) = 2$, while it is easy to see that

$$r_{fG}(f(x_j), f(y_j)) \longrightarrow \infty$$

as $j \to \infty$. In particular, $r_{fG}(f(x_j),f(y_j))$ has no upper bound in terms of $r_G(x_j,y_j)$. One can show that G is not a c-QED domain for any $c > 0$.

The function $r_G(x,y)$ is invariant under similarities and, accordingly, the same is true about 11.30 and 11.32. Next we shall give some $\mathcal{M}(\overline{\mathbf{R}}^n)$ –invariant results.

11.34. Theorem. *Let $D \subset \overline{\mathbf{R}}^n$ be a c-QED domain with* $\mathrm{card}(\overline{\mathbf{R}}^n \setminus D) \geq 2$ *and let $f\colon D \to fD \subset \overline{\mathbf{R}}^n$ be K–qc. Then for $x,y \in D$*

$$m_{fD}(f(x),f(y)) \leq \tau^{-1}\Big(\frac{c}{2^{n+1}K}\,\tau(m_D(x,y))\Big)$$

where m_D is as in (8.33).

Proof. The proof follows from 8.41 and 10.18(2). □

11.35. Theorem. *Let $f\colon \mathbf{B}^n \to \overline{\mathbf{R}}^n$ be a K–quasimeromorphic mapping, let $a,d \in \overline{\mathbf{R}}^n \setminus f\mathbf{B}^n$ be distinct and suppose that $N(f,\mathbf{B}^n) \leq p < \infty$. Then*

$$\frac{q(a,d)\,q(f(x),f(y))}{q(a,f(x))\,q(f(y),d)} \leq \tau^{-1}\Big(\frac{1}{8Kp}\,\tau\Big(\frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}\Big)\Big)$$

for all $x,y \in \mathbf{B}^n$.

Proof. By 8.6(2)

$$\lambda_{\mathbf{B}^n}(x,y) = \tfrac{1}{2}\,\tau\Big(\frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}\Big)$$

for distinct $x, y \in \mathbf{B}^n$. Let $D = \overline{\mathbf{R}}^n \setminus \{a, d\}$. By 8.5 $\lambda_{f\mathbf{B}^n} \le \lambda_D$ and thus by 8.40 we obtain

$$\lambda_{f\mathbf{B}^n}\big(f(x), f(y)\big) \le \lambda_D\big(f(x), f(y)\big) \le 4\,\tau\big(m_D(f(x), f(y)\big)$$
$$\le 4\,\tau(|\,a, f(x), d, f(y)\,|)\,.$$

The desired conclusion follows from the above inequalities and 10.18(2). ❐

11.36. Exercise. Show that $\operatorname{th}\frac{1}{2}\rho(x,y) = |x-y|/\sqrt{|x-y|^2 + 4x_n y_n}$ for $x, y \in \mathbf{H}^n$. Show that if $f\colon \mathbf{H}^n \to \mathbf{H}^n$ is K–qr with $f(e_n) = e_n$ then

$$\frac{|f(x)-e_n|}{\sqrt{|f(x)-e_n|^2 + 4f(x)_n}} \le \lambda_n^{1-\alpha}\Big[\frac{|x-e_n|}{\sqrt{|x-e_n|^2 + 4x_n}}\Big]^{\alpha}$$

for all $x \in \mathbf{H}^n$. [Hint: Apply 11.2(1).] Assume next that $f : \mathbf{H}^n \to \mathbf{H}^n$ is K–qc and $f(e_n) = e_n$. Show that

$$\frac{|f(x)-e_n|^2}{4f(x)_n} \le \tau^{-1}\Big(\frac{1}{K}\,\tau\Big(\frac{|x-e_n|^2}{4x_n}\Big)\Big)\,.$$

[Hint: Find first an expression for $\operatorname{sh}\frac{1}{2}\rho(x,y)$ and then apply 11.22.]

11.37. Exercise. (1) Let $f\colon \mathbf{B}^n \to \mathbf{B}^n$ be K–qr and $\alpha = K^{1/(1-n)}$. From the proof of 11.14 derive the inequality

$$|f(x) - f(y)| \le (2\lambda_n)^{1-\alpha}\,\rho(x,y)^{\alpha}$$

for all $x, y \in \mathbf{B}^n$.

(2) Next extend this inequality to a K–qm map $f\colon \mathbf{B}^n \to Q(z,r)$ where $Q(z,r)$ is a ball in the spherical metric as defined in (1.22). Show that

$$q(f(x), f(y)) \le (2\lambda_n)^{1-\alpha}\,\rho(x,y)^{\alpha}\,c(r)$$

for all $x, y \in \mathbf{B}^n$ where $c(r) = r/\sqrt{1-r^2}$.

(3) Find a form of 11.14 where the majorant is independent of n.

(4) Let $f\colon \mathbf{B}^n \to \mathbf{R}^n$ be a K–qr mapping with $B_f = \emptyset$. Show that for $n \ge 3$ there exists a number $d(n,K)$ such that for all $r \in (0,1)$

$$N(f, B^n(r)) \le d(n,K)(1-r)^{1-n}\,.$$

[Hint: 4.22 and 11.25.]

11.38. Exercise. Let $f\colon \mathbf{B}^n \to \mathbf{R}^n$ be K-qr and assume that there are numbers $T > 0$ and $A > 0$ such that $\rho(x,y) \le T$ implies $|f(x) - f(y)| \le A$. Show that

$$|f(x) - f(y)| \le A\,\lambda_n^{1-\alpha} \left[\frac{\operatorname{th} \frac{1}{2}\rho(x,y)}{\operatorname{th} \frac{1}{2}T} \right]^{\alpha}$$

for all $x, y \in \mathbf{B}^n$ with $\rho(x,y) \le T$ where $\alpha = K^{1/(1-n)}$. Next combine this inequality with 4.13 to obtain a bound valid for all $x, y \in \mathbf{B}^n$.

Next we shall survey some distortion theorems for quasiconformal mappings, which will not be proved in this book.

In 1956 the following theorem of A. Mori [MOR1] appeared.

11.39. Theorem. *Let $f\colon \mathbf{B}^2 \to \mathbf{B}^2$ be a K-qc mapping with $f(0) = 0$ and $f\mathbf{B}^2 = \mathbf{B}^2$. Then*

$$|f(x) - f(y)| \le 16\,|x - y|^{1/K}$$

for all $x, y \in \mathbf{B}^2$. Furthermore, the number 16 cannot be replaced by any smaller absolute constant.

It has been conjectured that the best constant in place of 16 is $16^{1-1/K}$ ([LV2, p. 68]). In 1985 H. Qu [Q] proved that the constant $16^{2(1-1/K)}$ will do (cf. [SEM2, p. 205]). In [FV] R. Fehlmann and M. Vuorinen proved the following theorem.

11.40. Theorem. *Let $f\colon \mathbf{B}^n \to \mathbf{B}^n$ be a K-qc mapping with $f(0) = 0$ and $f\mathbf{B}^n = \mathbf{B}^n$. Then*

$$|f(x) - f(y)| \le M_1(n,K)\,|x - y|^{\alpha}\,, \quad \alpha = K^{1/(1-n)}\,,$$

for all $x, y \in \mathbf{B}^n$, where the number $M_1(n,K)$ has the following three properties

(1) *$M_1(n,K) \to 1$ as $K \to 1$, uniformly in n.*

(2) *$M_1(n,K)$ remains bounded for fixed K and varying n.*

(3) *$M_1(n,K) \le 3\lambda_n^2$ for all $K \ge 1$.*

We remark that a multidimensional generalization of 11.39 (essentially part (3) of (11.40)) follows if one extends Mori's original argument to $\mathbf{R}^n$. This fact was observed by B. V. Shabat in 1960 [SH] (see also F. W. Gehring [G2, p. 387] and K. Ikoma [IK]). The point of 11.40 is that a quantitative constant is given which satisfies the property (1). See also G. D. Anderson and M. K. Vamanamurthy [AV].

Some related results are given by R. Näkki and B. Palka [NP] as well as by F. W. Gehring and O. Martio [GM2].

11.41. Remark. It is an open problem whether the constant $M_1(n,K)$ in 11.40 can be chosen so that it remains bounded when both $n \to \infty$ and $K \to \infty$.

The following theorem was proved by P. Tukia and J. Väisälä ([TV], [V11]) and in its present improved dimension–independent form by G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen [AVV1].

11.42. Theorem. *For* $K \geq 1$ *and* $s \in (0,1)$ *there exists a homeomorphism* $\eta\colon [0,\infty) \to [0,\infty)$ *with* $\eta(0) = 0$ *and with the following properties. If* $f\colon \mathbf{B}^n \to \mathbf{R}^n$, $n \geq 2$, *is a* K*-qc mapping into* $\mathbf{R}^n$ *and* $x,y,z \in \overline{B}^n(s)$ *with* $x \neq z$, *then*

$$\frac{|f(x)-f(y)|}{|f(x)-f(z)|} \leq \eta\left(\frac{|x-y|}{|x-z|}\right) .$$

11.43. Exercise. Show that the inequalities $\eta(1) \geq 1$ and

$$\frac{|f(x)-f(y)|}{|f(x)-f(z)|} \geq 1\Big/\eta\left(\frac{|x-z|}{|x-y|}\right)$$

for all distinct $x,y,z \in \overline{B}^n(s)$ follow from 11.42. Show also that $\eta(1)$ yields a bound for the linear dilatation of the mapping f.

11.44. An open problem. For $K \geq 1$, $n \geq 2$, and $r \in (0,1)$ let

$$\varphi^*_{K,n}(r) = \varphi^*_K(r) = \sup\{\, |f(x)| : f \in \mathcal{QC}_K(\mathbf{B}^n),\, f(0) = 0,\, |x| \leq r\,\}$$

where $\mathcal{QC}_K(\mathbf{B}^n) = \{\, f\colon \mathbf{B}^n \to f\mathbf{B}^n \mid f$ is K–qc and $f\mathbf{B}^n \subset \mathbf{B}^n \,\}$. As shown in [LV2, p. 64]

$$\varphi^*_{K,2}(r) = \varphi_{K,2}(r) \leq 4^{1-1/K}\, r^{1/K} \tag{11.45}$$

for each $r \in (0,1)$ and $K \geq 1$. By 11.3(1)

$$\varphi^*_{K,n}(r) \leq \varphi_{K,n}(r) \leq \lambda_n^{1-\alpha}\, r^{\alpha}\,, \quad \alpha = K^{1/(1-n)}\,, \tag{11.46}$$

for $n \geq 2$, $K \geq 1$, $r \in (0,1)$. A. V. Sychev [SY, p. 89] has conjectured that

$$\varphi^*_{K,n}(r) \leq 4^{1-\alpha}\, r^{\alpha} \tag{11.47}$$

for all $n \geq 2$ and $K \geq 1$. Because $\lambda_2 = 4$, (11.47) agrees with (11.45) for $n = 2$. In [AVV4] it is shown that $\varphi^*_{K,n} \not\equiv \varphi_{K,n}$ for $n \geq 3$. It follows from 11.19 and 11.23 that

$$\begin{cases} \varphi^*_{K,n}(r) \leq 4\, r^{1/K}\,, \\ \left[\varphi^*_{K,n}(r)\right]^2 \leq 1 - \varphi^2_{1/K,n}(\sqrt{1-r^2})\,. \end{cases} \tag{11.48}$$

From (11.48) and (11.46) it follows, as shown in [AVV2], that

$$\varphi^*_{K,n}(r) \le 4^{1-1/K^2} r^{1/K} \tag{11.49}$$

holds for all $n \ge 2$, $K \ge 1$, $r \in (0,1)$. Note that the right hand side of (11.49) is bounded when $K \to \infty$. Recall that $\lambda_n \to \infty$ as $n \to \infty$ by 7.22 and that $\lambda_n^{1-\alpha} \le 2^{1-1/K} K$ by 7.51. Note that Sychev's conjecture (11.47) still remains open.

11.50. Notes. Distortion theorems for qc and qr mappings have been proved by many authors (see the bibliography of [C1]). The Hölder continuity of plane qc mappings was proved by L. V. Ahlfors [A1], and the Schwarz lemma by J. Hersch and A. Pfluger [HEP] and P. P. Belinskiĭ (see the references in [BEL, p. 13]). For $n = 2$ the explicit bound $4^{1-1/K}$ in 11.3(1) was found by C.–F. Wang [WA] with the aid of a parametric method, and a simplified proof was given by O. Hübner [HÜ]. See also O. Lehto and K. I. Virtanen [LV2, p. 65, (3.6)] as well as P. P. Belinskiĭ [BEL, p. 15, formula (16')]. The n–dimensional form of the proof in [HÜ] and [LV2] was given by G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen [AVV1].

The Hölder continuity of qr mappings in $\mathbf{R}^n$ was proved by E. D. Callender [CAL], F. W. Gehring [G2], Yu. G. Reshetnyak [R1], [R12, pp. 36–38]. A spatial form of the Schwarz lemma was found by B. V. Shabat [SH] and O. Martio, S. Rickman, and J. Väisälä [MRV2]. Most of these bounds depend essentially on n, with bounds that approach ∞ as $n \to \infty$. Dimension–free bounds (such as 11.3(1)) were given in [AVV1] and [AVV2]. For 11.27 see [VU10] and [MI1]. Both 11.34 and 11.35 were proved in [VU13]. For results similar to 11.34 and 11.35 see [G7, p. 233] and [RI2].

12. Uniform continuity properties

The present section is devoted to the study of uniform continuity properties of a qr mapping $f\colon G \to fG$ as a mapping between the metric spaces (G, k_G) and (fG, k_{fG}) and to the study of its restrictions

$$f|D : (D, k_D) \longrightarrow (fD, k_{fD})$$

whenever D is a subdomain of G. We shall consider the modulus of continuity of f (cf. (10.20))

$$\omega_f(t) = \sup\{\, k_{fG}(f(x), f(y)) : k_G(x,y) \le t \,\} .$$

If f is a Möbius transformation, then $\omega_{f|D}(t) \le 2t$ for all domains D in G by 3.10. In this section we shall prove an analogous result for qc maps. The situation for non-homeomorphic qr mappings is entirely different, as Example 11.3 in the preceding section shows. However, under a natural additional condition, one can prove a positive result even for non-homeomorphic mappings. This additional condition is a necessary and sufficient condition for a qr mapping $f\colon (G, k_G) \to (fG, k_{fG})$ to be uniformly continuous. The condition requires that the function d_f defined by

$$x \longmapsto d_f(x) = d(f(x), \partial G')$$

satisfy the Harnack inequality (4.11) in G. Applying some results of Section 8 we shall show that this Harnack condition is satisfied if f is qc or if f is qr and, in addition, $N(f,G) < \infty$. Furthermore, under mild restrictions on ∂fG the Harnack condition holds independently of $N(f,G)$. For instance, it is sufficient to require ∂fG to be connected.

We shall first prove some preliminary results.

12.1. Lemma. *Let* $R > 0$, $u, v \in \mathbf{R}^n \setminus B^n(R)$, $u \ne v$, *and let* F *be a continuum with* $u, v \in F$. *Then*

$$\mathsf{M}\left(\Delta(F,\, S^{n-1}(R))\right) \ge \gamma(1 + a(u,v))$$

where

$$a(u,v) = \frac{2\min\{\,|v|(|u|-R)\,,\, |u|(|v|-R)\,\}}{R\,|u-v|}\ .$$

Proof. Let $h(x) = Rx/|x|^2$, $|x| > R$. Then $h(\mathbf{R}^n \setminus \overline{B}^n(R)) = \mathbf{B}^n$. By (1.5)

$$|h(u) - h(v)| = \frac{|u-v|}{|u||v|}\, R\ .$$

This together with the definition (2.34) yields

$$j_{\mathbf{B}^n}\big(h(u), h(v)\big) = \log(1 + 2/a(u,v))\ .$$

By conformal invariance 5.17, 7.32, and 2.41(1)

$$\mathsf{M}\big(\Delta(F,\, S^{n-1}(R))\big) = \mathsf{M}\big(\Delta(h(F),\, S^{n-1})\big) \ge \gamma\Big(\frac{1}{\operatorname{th}\frac{1}{2}\rho(h(u), h(v))}\Big)$$

$$\ge \gamma\Big(\frac{1}{\operatorname{th}(\frac{1}{2}\, j_{\mathbf{B}^n}(h(u), h(v)))}\Big)\ .$$

Because $\operatorname{th}\big(\frac{1}{2}\log(1+s)\big) = s/(2+s)$, we obtain

$$\mathsf{M}\big(\Delta(F,\, S^{n-1}(R))\big) \ge \gamma(1 + a(u,v))$$

as desired. □

12.2. Lemma. *Let* $f\colon G \to \mathbf{R}^n$ *be a qr mapping, let* G *and* fG *be proper subdomains of* $\mathbf{R}^n$, $x \in G$, $\theta \in (0, \frac{1}{2})$, *and let* $z \in \partial fG$ *with*

$$d_f(x) = |f(x) - z| = d(f(x), \partial fG) \, .$$

Assume that $|x - y| < \frac{1}{2}\, d(x)$ *implies* $|f(y) - z| \geq \theta\, d_f(x)$. *Then the inequality*

$$\frac{|f(x) - f(y)|}{d_f(x)} \leq \frac{A}{\gamma^{-1}(K\,\gamma(d(x)/(2|x-y|))\,) - A - 1}$$

holds for $|x - y| < \frac{1}{2}\, d(x)$, *where* $K = K_I(f)$ *and* $A = 2(\theta^{-1} - 1)$.

Proof. Let $B_x = B^n(x, \frac{1}{2}d(x))$. We may assume that $f(x) \neq f(y)$. By the monotone property 8.5 of μ_G , 10.18(1), and 8.8(2),

$$\begin{aligned}\mu_{fG}\big(f(x), f(y)\big) &\leq K\,\mu_G(x,y) \leq K\,\mu_{B_x}(x,y)\\ &\leq K\,\gamma\Big(\frac{d(x)}{2\,|x-y|}\Big)\,,\end{aligned}$$

where $K = K_I(f)$. Next apply 8.5 and 12.1 with $R = \theta\, d_f(x)$ to get

$$\mu_{fG}\big(f(x), f(y)\big) \geq \gamma(1 + a)$$

where

$$a = \frac{2\min\{\,|f(y) - z|(|f(x) - z| - R)\,,\ |f(x) - z|(|f(y) - z| - R)\,\}}{R\,|f(x) - f(y)|}\,.$$

Since $|f(y) - z| \leq |f(x) - f(y)| + |f(x) - z|$ and $R = \theta\, d_f(x)$ we obtain

$$a \leq 2(\theta^{-1} - 1)\Big(1 + \frac{d_f(x)}{|f(x) - f(y)|}\Big)\,.$$

This inequality together with the above ones yields

$$\gamma\Big(1 + 2(\theta^{-1} - 1)\Big(1 + \frac{d_f(x)}{|f(x) - f(y)|}\Big)\Big) \leq K\,\gamma\Big(\frac{d(x)}{2|x-y|}\Big)\,.$$

The desired inequality is now easily obtained from this. ❐

12.3. Corollary. *Under the assumptions of 12.2, there exists a number* $t_0 \in (0, \frac{1}{2})$ *depending only on* $K_I(f)$ *and* θ *such that* $|x - y| \leq t_0\, d(x)$ *implies*

$$\frac{|f(x) - f(y)|}{d_f(x)} \leq \tfrac{1}{2}\,.$$

Proof. Denote $K = K_I(f)$. As in (7.44) let

$$\varphi_K(r) = \frac{1}{\gamma^{-1}(K\gamma(1/r))} \, , \quad r \in (0,1) \, .$$

We can now rewrite the inequality of 12.2 as

$$\frac{|f(x) - f(y)|}{d_f(x)} \le \frac{A\,\varphi_K(2|x-y|/d(x))}{1-(1+A)\varphi_K(2|x-y|/d(x))} = B$$

where $A = 2(\theta^{-1} - 1)$. Hence it suffices to require $B = \frac{1}{2}$, in other words

$$(3A+1)\,\varphi_K\big(2|x-y|/d(x)\big) = (6\theta^{-1} - 5)\,\varphi_K\big(2|x-y|/d(x)\big) = 1 \, .$$

In order to find a number t_0 independent of the dimension we recall that by 7.47(1) and 7.50

$$\varphi_{K,n}(t) \le 2^{1-1/K} K\, t^{1/K}$$

holds for $K \ge 1$, $n \ge 2$, and $t \in (0,1)$. Hence it suffices to choose t_0 so that

$$(6\theta^{-1} - 5)\, 2^{1-1/K} K (2t_0)^{1/K} = 1$$

or, equivalently,

$$t_0 = (12K\theta^{-1} - 10\,K)^{-K} \, .$$

Because $\theta \in (0, \frac{1}{2})$ we see that

(12.4) $$\left(\frac{\theta}{12K}\right)^K \le t_0 \le \left(\frac{\theta}{7K}\right)^K \, .$$

Hence t_0 depends only on θ and K as desired. ☐

After these auxiliary results we now prove the main result of this section.

12.5. Theorem. *For $K \ge 1$ and $\theta \in (0, \frac{1}{2})$ there exists a number c with the following property. Let G and G' be proper subdomains of $\mathbf{R}^n$ and let $f\colon G \to \mathbf{R}^n$ be a non-constant qr mapping with $fG \subset G'$ satisfying the Harnack condition*

$$d(f(x), \partial G') \ge \theta\, d(f(y), \partial G')$$

for all $x, y \in G$ with $|x-y| \le \frac{1}{2}\, d(x)$. If $K_I(f) = K$ and $\alpha = K^{1/(1-n)}$, then

$$k_{G'}\big(f(x), f(y)\big) \le c \max\{\, k_G(x,y)^\alpha, \; k_G(x,y) \,\}$$

for all $x, y \in G$.

Outline of proof. The proof will be carried out in two steps. In the first step we choose a number $t \in (0, \frac{1}{2})$, $t = t(K, \theta)$, such that $|x - y| \le t\, d(x)$ implies $|f(x) - f(y)| < \frac{1}{2}\, d(f(x), \partial G')$ whenever $x \in G$. Moreover, we prove the theorem for $|x - y| \le t\, d(x)$. In the second step we assume that $|x - y| \ge t\, d(x)$ and prove the theorem in this case by exploiting quasihyperbolic geodesics as in Lemma 4.9.

Proof of 12.5. Fix $x, y \in G$ with $y \in B^n(x, \frac{1}{2}d(x)) = B_x$. Choose $z_0 \in \partial G'$ such that $|f(x) - z_0| = d(f(x), \partial G') = d(f(x))$. From the Harnack condition it follows that f maps B_x into $\mathbf{R}^n \setminus \overline{B}^n(z_0, \theta\, d(f(x)))$. Let $t = t(K, \theta)$ be the number given by 12.3. Because $\theta \in (0, \frac{1}{2})$ we obtain by (12.4)

$$\left(\frac{\theta}{12K}\right)^K \le t \le \left(\frac{\theta}{7K}\right)^K . \tag{12.6}$$

Then

$$\frac{|f(x) - f(y)|}{d(f(x))} \le \tfrac{1}{2} \tag{12.7}$$

for $|x - y| \le t\, d(x)$.

Case A. $|x - y| \le t\, d(x)$. Let $B_1 = B^n(x,\, td(x))$ and $B_2 = B^n(f(x), \frac{1}{2}d(f(x)))$. For $y \in B_1$ we obtain by 8.8(2)

$$\mu_{B_1}(x, y) = \gamma\left(\frac{t\, d(x)}{|x - y|}\right) .$$

Observe that $fB_1 \subset B_2$ by (12.7). Hence the monotone property 8.5 of μ_G together with 8.8(2) yield

$$\mu_{fB_1}(f(x), f(y)) \ge \mu_{B_2}(f(x), f(y)) = \gamma\left(\frac{d(f(x))}{2|f(x) - f(y)|}\right) .$$

Because

$$\mu_{fB_1}(f(x), f(y)) \le K\, \mu_{B_2}(x, y)$$

by 10.18(1), the above relations yield

$$\frac{2|f(x) - f(y)|}{d(f(x))} \le \frac{1}{\gamma^{-1}(K\, \gamma(td(x)/|x - y|))} = \varphi_K\left(\frac{|x - y|}{t\, d(x)}\right) \tag{12.8}$$

where we have used the function φ_K introduced in (7.44). Because $|x - y| \le t\, d(x)$, also (12.7) holds, and hence by 3.7(1)

$$k_{G'}(f(x), f(y)) \le \log\left(1 + \frac{|f(x) - f(y)|}{d(f(x)) - |f(x) - f(y)|}\right) \le \frac{2|f(x) - f(y)|}{d(f(x))} .$$

This inequality together with (12.8), 7.47, and 7.50 yields

$$k_{G'}(f(x), f(y)) \le 2^{1-1/K} K \Big(\frac{|x-y|}{t\, d(x)}\Big)^{\alpha} \tag{12.9}$$

where $\alpha = K^{1/(1-n)}$. It follows from 3.7(1) and (12.6) that

$$k_G(x,y) \le \log\Big(1 + \frac{|x-y|}{d(x)(1-t)}\Big) \le \log \frac{1}{1-t} \le \log \frac{7}{6} < \frac{1}{6} \,.$$

Therefore we have by (3.4) for $|x-y| \le t\, d(x)$

$$\frac{|x-y|}{d(x)} \le \exp\Big(k_G(x,y)\Big) - 1 \le \frac{k_G(x,y)}{1-1/6} = \frac{5}{6}\, k_G(x,y) \,. \tag{12.10}$$

Here the second inequality follows from the well-known fact that [AS, 4.2.33]

$$e^a - 1 < \frac{a}{1-a}$$

for $a < 1$. By (12.9), (12.10), and (12.6) we obtain

$$k_{G'}(f(x), f(y)) \le c_A\, k_G(x,y)^{\alpha} \tag{12.11}$$

where

$$c_A = 2^{1-1/K} K (5/6)^{\alpha} (12K/\theta)^{K\alpha} \le 2^{1-1/K} K (12K/\theta)^K \,. \tag{12.12}$$

Case B. $|x-y| > t\, d(x)$. Let $J_G[x,y] = J$ be a geodesic segment of the quasi-hyperbolic metric k_G . Choose points $x_1, \ldots, x_{p+1}$ on J as follows. Let $x_1 = x$ and assume that the points $x_1, \ldots, x_j$ have been chosen. If $y \in \overline{B}^n(x_j, t\, d(x_j))$ we set $p = j$, $x_{p+1} = y$ and the process of choosing points ends. Otherwise we choose x_{j+1} to be the last point of J on $S^{n-1}(x_j, t\, d(x_j))$ when we traverse from x to y along the geodesic segment J . It follows from (3.4) that

$$k_G(x_j, x_{j+1}) \ge \log(1+t)$$

for $1 \le j \le p-1$. By the length-minimizing property of the geodesic J

$$(p-1)\log(1-t) \le \sum_{j=1}^{p-1} k_G(x_j, x_{j+1}) \le k_G(x_1, x_{p+1}) = k_G(x,y)$$

and hence $p \le 1 + k_G(x,y)/\log(1+t)$. By the definition of the number t (see (12.6)) we see by 3.7(1) that

$$k_{G'}(f(x_j), f(x_{j+1})) \le \log 2$$

for all $j = 1, \ldots, p$. Therefore by the triangle inequality for $k_{G'}$ and by (3.4) we obtain the desired inequality

$$k_{G'}(f(x), f(y)) \leq p \log 2 \leq (\log 2)\Big(1 + \frac{k_G(x,y)}{\log(1+t)}\Big) \leq (\log 4)\,\frac{k_G(x,y)}{\log(1+t)}\,,$$

because $|x-y| \geq t\,d(x)$ in the Case B. By (12.6) the constant admits the following upper bound

$$c_B = \frac{\log 4}{\log(1+t)} \leq \frac{(12K/\theta)^K \log 4}{7\log(8/7)} \leq \frac{3}{2}(12K/\theta)^K\,. \tag{12.13}$$

Finally, by (12.12) and (12.13) we see that in both Cases A and B we can choose

$$c = \max\{\, c_A,\, c_B \,\} < 3 \cdot 2^{-1/K} K (12K/\theta)^K\,. \quad \square \tag{12.14}$$

12.15. Exercise. In the above computations we applied the fact that in view of (12.6), $t \leq 1/7$. However, it was required in 12.5 that $\theta \leq \frac{1}{2}$ and hence $t \leq 1/14$ by (12.6). Using this fact improve the constant c in (12.14).

12.16. Corollary. *Let* $f\colon G \to \mathbf{R}^n$ *be a non-constant qr mapping such that* $fG \subset G'$. *Then*

$$f : (G, k_G) \longrightarrow (G', k_{G'})$$

is uniformly continuous if and only if the Harnack condition of 12.5 *holds.*

Proof. By 12.5 it will be enough to prove that uniform continuity implies the Harnack condition. Assume that f is uniformly continuous in the above sense. Hence there exists a number D such that $k_G(x,y) \leq \log(3/2)$ implies $k_{G'}(f(x), f(y)) \leq D$. It follows from (3.4) that $|x-y| \leq \frac{1}{2}d(x)$ for $k_G(x,y) \leq \log(3/2)$. Hence for $|x-y| \leq \frac{1}{2}d(x)$ we obtain by (3.5)

$$\Big|\log \frac{d(f(x))}{d(f(y))}\Big| \leq k_{G'}(f(x), f(y)) \leq D$$

where $d(f(x)) = d(f(x), \partial G')$. Thus the Harnack condition of 12.5 is fulfilled with $\theta = e^{-D}$. $\square$

We next show that p–to–one qr mappings satisfy the Harnack condition of 12.5.

12.17. Theorem. *Let G and G' be proper subdomains of $\mathbf{R}^n$ and let $f\colon G \to \mathbf{R}^n$ be a qr mapping with $fG \subset G'$ and $N(f,G) \le p < \infty$. Then for all $x, y \in G$ with $|x-y| \le \frac{1}{2} d(x)$*

$$d(f(x)) \le \left[1 + \tau^{-1}(A\,\tau(1/24))\right] d(f(y))$$

where $d(f(x)) = d(f(x), \partial G')$ and $A = 1/(8pK_O(f))$.

Proof. We may assume that $d(f(x)) > d(f(y))$. Because

$$\frac{|f(x)-f(y)|}{\min\{\, d(f(x))\,,\, d(f(y))\,\}} \ge \frac{d(f(x))}{d(f(y))} - 1$$

by the triangle inequality, Corollary 8.25 yields

$$\lambda_{G'}(f(x), f(y)) \le 4\,\tau\Big(\frac{d(f(x))}{d(f(y))} - 1\Big)\,.$$

It follows from 8.6(2) and (2.17) that

$$\lambda_{\mathbf{B}^n}(0,z) \ge \tfrac{1}{2}\,\tau\big(\mathrm{sh}^2(\tfrac{1}{2}\log\tfrac{3}{2})\big) \ge \tfrac{1}{2}\,\tau(1/24)$$

for all $|z| \le \frac{1}{2}$. Denote $B_x = B^n(x, \frac{1}{2}d(x))$. Then

$$\lambda_G(x,y) \ge \lambda_{B_x}(x,y) \ge \tfrac{1}{2}\,\tau(1/24)$$

by 8.5 and the above inequality. The desired inequality follows now from 10.18(2). □

12.18. Exercise. Applying the functional identity $\gamma(t) = 2^{n-1}\tau(t^2-1)$ of 5.53 show that

$$1 + \tau^{-1}(M\tau(t)) = \left[\gamma^{-1}\big(M\gamma(\sqrt{1+t}\,)\big)\right]^2$$

for all $M > 0$ and $t > 0$. Next show that the constant in 12.17 has an upper bound in terms of $pK_O(f)$. [Hint: Apply 7.51.]

12.19. Corollary. *Let $f\colon G \to fG$ be a qc mapping where G and fG are proper subdomains of $\mathbf{R}^n$. Then*

$$k_{fG}(f(x), f(y)) \le c\max\{\, k_G(x,y)^{\alpha}\,,\, k_G(x,y)\,\}$$

holds for all $x, y \in G$ where $\alpha = K_I(f)^{1/(1-n)}$ and c depends only on $K_O(f)$.

Proof. By 12.17 and 12.18 the Harnack condition of 12.5 holds with a dimension–free constant θ_0. The proof follows now from 12.5. □

12.20. Corollary. *Let* $f\colon G \to fG$ *be a* K*–qc mapping, where* G *and* fG *are proper subdomains of* $\mathbf{R}^n$. *Then*

$$k_{fG}\big(f(x), f(y)\big) \le c_1 \max\{\, k_G(x,y)^{1/K},\, k_G(x,y)\,\}$$

holds for all $x, y \in G$ *where* c_1 *depends only on* K.

Proof. Because $K \ge K_O(f)$ and because the constant c of 12.19 increases with $K_O(f)$ we can make c independent of $K_O(f)$ by replacing $K_O(f)$ with K. This yields a new constant c_1 depending only on K with $c_1 \ge c$. Because $\alpha = K_I(f)^{1/(1-n)} \ge 1/K$ we obtain

$$\max\{\, k_G(x,y)^{\alpha},\, k_G(x,y)\,\} \le k_G(x,y)^{1/K}$$

for $k_G(x,y) \le 1$ and

$$\max\{\, k_G(x,y)^{\alpha},\, k_G(x,y)\,\} = k_G(x,y)$$

for $k_G(x,y) \ge 1$. The desired dimension–free inequality follows. □

It follows from Example 11.4 that Corollary 12.19 does not hold for qr mappings and not even for analytic functions. However, if ∂fG satisfies some additional conditions, then 12.19 can be generalized to qr mappings. Next we shall prove such a result when ∂fG is connected.

12.21. Theorem. *Let* $f\colon G \to \mathbf{R}^n$ *be a non–constant qr mapping and let* ∂fG *be a continuum containing at least two distinct points. Then*

$$k_{fG}\big(f(x), f(y)\big) \le c_2 \max\{\, k_G(x,y)^{\alpha},\, k_G(x,y)\,\}$$

for all $x, y \in G$ *where* c_2 *depends only on* n *and* $K_I(f)$.

Proof. Let $x, y \in G$ with $|x-y| \le \frac{1}{2} d(x)$. By 10.18(1) and 8.8(2) we obtain

$$\mu_{fG}\big(f(x), f(y)\big) \le K_I(f)\, \mu_G(x,y) \le K_I(f)\, \gamma(2)\,.$$

Further, in view of (3.5) and 8.31(1)

$$c_n \left| \log \frac{d(f(x))}{d(f(y))} \right| \le \mu_{fG}\big(f(x), f(y)\big)\,.$$

From these inequalities it follows that $d(f(x)) = d(f(x), \partial fG)$ satisfies the Harnack condition of 12.5 with

$$\theta = \exp\bigl(-K_I(f)\,\gamma(2)/c_n\bigr) \ .$$

Hence the proof follows from 12.5. ❒

It should be observed that Theorem 12.21 is applicable to qr mappings also when $N(f,G) = \infty$.

12.22. Theorem. *Let* $f\colon G \to \mathbf{R}^n$ *be a non-constant* K*-qr mapping and let* $D \in J(G)$ (for notation see 9.10). *Then*

$$k_{fD}\bigl(f(x), f(y)\bigr) \le c(D) \max\{\, k_D(x,y)^\alpha,\ k_D(x,y)\,\}$$

holds for all $x, y \in D$ *where* $\alpha = K^{1/(1-n)}$ *and* $c(D)$ *depends only on* $KN(f,D)$.

Proof. It follows from 12.17 and 12.18 that $d(f(x), \partial fD)$ satisfies the Harnack condition of 12.5 in D with a dimension-free constant θ depending only on $KN(f,D)$. The proof follows now from 12.5. ❒

12.23. Example. For $n = 2$ and $K = 1$ consider the analytic functions $f_p(z) = z^p$, $p = 2, 3, \ldots$, $z \in \mathbf{C}$. The points $a_p = \frac{1}{2}$ and $b_p = \frac{1}{2}\exp[(\pi/p)i]$ are mapped by f_p onto $a'_p = 2^{-p}$ and $b'_p = -2^{-p}$, respectively. Let $D = \mathbf{B}^2 \setminus \{0\} = f_p D$. By (3.4)

$$k_{f_pD}(a'_p, b'_p) \ge j_D(a'_p, b'_p) = \log 3 > 1 \ ,$$
$$k_D(a_p, b_p) \le \pi/p \ ,$$

where the last inequality follows by integration along the circular arc $\{z \in \mathbf{C} : z = \frac{1}{2}\exp(ti),\ 0 \le t \le \pi/p\}$ (see the definition (3.2) of the quasihyperbolic metric). By Theorem 12.22

$$1 < \log 3 \le c(D) \max\{(\pi/p)^1, \pi/p\} = c(D)\pi/p$$

and hence

$$c(D) \ge p/\pi \ge N(f_p, D)/\pi \ .$$

In particular, we see that $c(D) \to \infty$ as $N(f,D) \to \infty$ in 12.22.

12.24. Corollary. *Let* $f\colon \mathbf{B}^n \to Y$, $Y = \mathbf{R}^n \setminus \{0\}$, *be a qr mapping with* $N(f, \mathbf{B}^n) < \infty$. *Then*

$$f : (\mathbf{B}^n, \rho) \longrightarrow (Y, k_Y)$$

is uniformly continuous. In particular,

$$f : (\mathbf{B}^n, \rho) \longrightarrow (\mathbf{R}^n, q)$$

is uniformly continuous.

Proof. Theorem 11.24 shows that the Harnack condition of 12.5 is fulfilled and hence the first assertion follows from 12.5. The second assertion follows from the first one (see 3.31). □

12.25. Exercise. Show that 12.20 yields a bound for the linear dilatation of a K-qc mapping. [Hint: Apply 12.20 to $G \setminus \{x\}$, $x \in G$.]

12.26. Remark. (1) Let $\overline{c}$ denote the least constant with which 12.19 holds. As shown in [AVV2] the following inequalities hold

$$\left(1 + \frac{1}{\pi^2} \log^2 \lambda(K)\right)^{1/2} \le \overline{c} \le 2K\left[1 + \left(2(\sqrt{3} + \sqrt{2})\right)^{8K}\right],$$

where $\lambda(K)$ is as in 10.31 and $K = K(f)$.

(2) The condition in 12.21 that ∂fG be a non-degenerate continuum can be replaced by the requirement that ∂fG be sufficiently thick at each of its points in a sense involving n-capacity. See [VU12].

(3) This section is taken from [VU10]. Corollary 12.19 is due to F. W. Gehring and B. G. Osgood [GOS].

13. Normal quasiregular mappings

The properties of bounded analytic functions of the unit disc have been studied extensively in classical function theory. In their fundamental paper [LV1] of 1957 O. Lehto and K. I. Virtanen proved that many boundary properties of bounded analytic functions, or more generally of meromorphic functions omitting at least three distinct values in the extended complex plane, have natural generalizations to a wider subclass of meromorphic functions, namely the normal meromorphic functions. This class of functions is very convenient to study because of its invariance properties. The notion of a normal meromorphic function also provides a natural setup for the study of the Schwarz lemma and the Schottky theorem as well as their many ramifications.

For a bibliography of normal meromorphic functions the reader is referred to A. J. Lohwater's survey [LOH] (see also [PO]).

The goal of this section is the study of some growth properties of normal quasiregular mappings. In the case of meromorphic functions there are several equivalent characterizations of normal functions, of which we mention here only three: namely one based on the study of normal families, one based on the notion of the spherical derivative, and finally one making use of uniform continuity between appropriate metric spaces. Of these the last one seems to be the most natural definition in the present context, since we are interested not only in knowing whether a function is normal but also in estimating its modulus of continuity.

13.1. Definition. A continuous mapping $f\colon \mathbf{B}^n \to \overline{\mathbf{R}}^n$ is said to be *normal* if $\omega_f(t) \to 0$ as $t \to 0$ where

$$\omega_f(t) = \sup\{\, q\big(f(x), f(y)\big) : x, y \in \mathbf{B}^n \text{ and } \rho(x,y) \le t \,\} .$$

Then $q\big(f(x), f(y)\big) \le \omega_f\big(\rho(x,y)\big)$ holds for all $x, y \in \mathbf{B}^n$ by virtue of the above definition. In other words, $f\colon \mathbf{B}^n \to \overline{\mathbf{R}}^n$ is normal if and only if f is uniformly continuous as a mapping between metric spaces $f\colon (\mathbf{B}^n, \rho) \to (\overline{\mathbf{R}}^n, q)$.

13.2. Remarks. (1) Since the hyperbolic metric (2.21) is a conformal invariant it follows that f is normal if and only if $f \circ g$ is normal for each $g \in \mathcal{M}(\mathbf{B}^n)$. In this case $\omega_f = \omega_{f\circ g}$. If $f\colon \mathbf{B}^n \to \overline{\mathbf{R}}^n$ is normal and $h \in \mathcal{M}(\overline{\mathbf{R}}^n)$, then so is $h \circ f$ and $\omega_{h\circ f} \le \mathrm{Lip}(h)\,\omega_f$ where $\mathrm{Lip}(h)$ is the Lipschitz constant in the spherical metric. In particular, $\omega_{h\circ f} = \omega_f$ if h is a spherical isometry. It follows that in most cases we may assume that $f(0) = 0$, by considering $t_{f(0)} \circ f$ in place of f , where $t_{f(0)}$ is the spherical isometry defined in (1.46). As we shall see in 13.4, from these invariance properties and from the Schwarz lemma 11.2 it follows that $\omega_f(t) \le c\,t^{\alpha}$, $\alpha = K^{1/(1-n)}$, if $f\colon \mathbf{B}^n \to \overline{\mathbf{R}}^n$ is K-qm and normal.

(2) The above definition of normality extends immediately to the case of functions defined in an arbitrary proper subdomain G of $\mathbf{R}^n$, if we use k_G in place of ρ . It should be observed that if G is a multiply connected plane domain then this definition of a normal function is not the same as the definition in [LV1].

13.3. Exercise. Assume that $f\colon \mathbf{B}^n \to \overline{\mathbf{R}}^n$ is normal and $f_1 \in \mathcal{M}(\mathbf{R}^n)$ with $f_1(x) = x + a$. Show that $\omega_{f_1\circ f}(t) \le \big[1 + \frac{1}{2}|a|\big(|a| + \sqrt{4 + |a|^2}\,\big)\big]\omega_f(t)$. [Hint: 1.54(4).]

Assume $f_1(x) = a + r^2(x-a)/|x-a|^2$. Find an upper bound for $\omega_{f_1 \circ f}(t)$. [Hint: 1.54(2).]

By virtue of the results in the preceding sections we see that the class of normal qm mappings is wide. Several examples will be given in 13.7.

The following result may be viewed as a generalization of the Schwarz lemma to the context of normal functions.

13.4. Theorem. *Let $f\colon \mathbf{B}^n \to \overline{\mathbf{R}}^n$ be a normal qm mapping and let $M = \sup\{ t : \omega_f(t) = \frac{1}{2} \}$. Then*

$$q(f(x), f(y)) \le a(n,K)\Big(\frac{\operatorname{th}\frac{1}{2}\rho(x,y)}{\operatorname{th}\frac{1}{2}M}\Big)^{\alpha} , \quad \alpha = K_I(f)^{1/(1-n)} ,$$

for all $x, y \in \mathbf{B}^n$ where $a(n,K) = \max\{1, \lambda_n^{1-\alpha}/\sqrt{3}\}$ and λ_n is the Grötzsch constant (7.24).

Proof. Since $a(n,K) \ge 1$, the assertion is trivial if $\rho(x,y) \ge M$. Fix $x, y \in \mathbf{B}^n$ with $\rho(x,y) < M$. In view of the conformal invariance of the right side, we may assume that $x = 0$ and $y = (\operatorname{th}\frac{1}{2}\rho(x,y))e_1$ (see (2.25)). Because the left side is invariant under spherical isometries we may also assume $f(0) = 0$. Hence $fD(0,M) \subset Q(0,\frac{1}{2}) = B^n(1/\sqrt{3})$ by 1.25(1). Denote

$$h_1 : \mathbf{B}^n \longrightarrow D(0,M) , \quad h_1(x) = x \operatorname{th}\tfrac{1}{2}M ,$$
$$h_2 : B^n(\sqrt{3}) \longrightarrow \mathbf{B}^n , \quad h_2(x) = x\sqrt{3} .$$

Then $g = h_2 \circ f \circ h_1 \colon \mathbf{B}^n \to \mathbf{B}^n$ is qr with $K_I(g) = K_I(f)$, $K_O(g) = K_O(f)$, $g(0) = 0$ and thus by 11.3(1) and 1.17

$$\sqrt{3}\, q(f(y), f(0)) \le \sqrt{3}\,|f(y) - f(0)| \le |g(y/\operatorname{th}\tfrac{1}{2}M) - g(0)|$$
$$\le \lambda_n^{1-\alpha}\Big(\frac{\operatorname{th}\frac{1}{2}\rho(x,y)}{\operatorname{th}\frac{1}{2}M}\Big)^{\alpha}$$

where $\alpha = K_I(f)$. This is the desired inequality. □

13.5. Corollary. *A qm mapping $f\colon \mathbf{B}^n \to \overline{\mathbf{R}}^n$ is normal if and only if there are numbers $\alpha \in (0,1]$ and $\beta > 0$ such that $\omega_f(t) \le \beta t^{\alpha}$ for all $t \in (0,\infty)$.*

If $f\colon \mathbf{B}^n \to \overline{\mathbf{R}}^n$ is continuous, then the set $E_t = \{ z \in \mathbf{B}^n : |f(z)| = t \}$, $t > 0$, is called the *t-level set* of $|f|$. We are next going to give a geometric characterization of a normal qr mapping which requires that the oscillation of the mapping "near" a level set is bounded. It should be observed that the hypothesis $S^{n-1} \cap f\mathbf{B}^n \neq \emptyset$ in the following theorem is merely a technical normalization: if it fails to hold, then f omits a ball of $\overline{\mathbf{R}}^n$ of spherical diameter $= 1$ and hence f will be normal by virtue of 11.1.

13.6. Theorem. *Let $f\colon \mathbf{B}^n \to \mathbf{R}^n$ be a non-constant qm mapping with $S^{n-1} \cap f\mathbf{B}^n \neq \emptyset$ and let $E = \{ z \in \mathbf{B}^n : |f(z)| \le 1 \}$. Then the following conditions are equivalent:*

(1) *f is normal.*

(2) *There exists a positive number T such that $|f(z)| \le e$ whenever $z \in \mathbf{B}^n \setminus E$ and $\rho(z,E) \le T$.*

Proof. Since the implication (1) $\Rightarrow$ (2) is obvious, only (2) $\Rightarrow$ (1) remains to be proved. Fix $x, y \in \mathbf{B}^n$ with $\rho(x,y) < \frac{1}{2}T$. We consider two cases.

Case 1. $\rho(x,E) \le \frac{1}{2}T$. In this case $f|D(x,\frac{1}{2}T)$ omits the set $F_1 = \overline{\mathbf{R}}^n \setminus B^n(e)$ by the hypothesis of the theorem and

$$c(F_1) \ge \frac{c(F_1,\infty)}{d_1} = \frac{\omega_{n-1}}{d_1}\big(\log(e\sqrt{3}\,)\big)^{1-n} = \bar{d}_1$$

by (6.13) and 6.14.

Case 2. $\rho(x,E) > \frac{1}{2}T$. In this case $f|D(x,\frac{1}{2}T)$ omits $\overline{\mathbf{B}}^n$ (i.e. $fD(x,\frac{1}{2}T) \cap \overline{\mathbf{B}}^n = \emptyset$) and

$$c(\overline{\mathbf{B}}^n) \ge \frac{c\big(\overline{\mathbf{B}}^n,\infty\big)}{d_1} = \frac{\omega_{n-1}}{d_1}(\log\sqrt{3}\,)^{1-n} = \bar{d}_2\ .$$

In both cases we apply 10.18(1) to $f|D(x,\frac{1}{2}T)$, and we obtain by (2.24) and 8.8(2)

$$\mu_{D(x,T/2)}(x,y) = \gamma\Big(\frac{\operatorname{th}\frac{1}{4}T}{\operatorname{th}\frac{1}{2}\rho(x,y)}\Big)\ .$$

Because $\bar{d}_1 < \bar{d}_2$ we obtain by 6.1 in both Cases 1 and 2

$$\begin{aligned}\mu_{fD(x,T/2)}\big(f(x),f(y)\big) &\ge \beta\min\{\,\bar{d}_1\,,\, d_3\, q\big(f(x),f(y)\big)\,\}\\ &\ge \beta\min\{\,\bar{d}_1\,,\, d_3\}\, q\big(f(x),f(y)\big)\ .\end{aligned}$$

This together with the previous inequality, 7.26(1), and 10.18(1) shows that f is normal. ❐

13.7. Examples. We now list some sufficient conditions for a qr mapping $f\colon \mathbf{B}^n \to \mathbf{R}^n$ to be normal:

(1) $c(\overline{\mathbf{R}}^n \setminus f\mathbf{B}^n) > 0$ (see 11.1). In particular, an injective qr mapping of $\mathbf{B}^n$ (i.e. qc mapping) is normal, because $c(\partial f\mathbf{B}^n) > 0$ by 14.6(1) and 6.1. Likewise, bounded qr maps are normal.

(2) $f\mathbf{B}^n \subset G$, where G is a proper subdomain of $\mathbf{R}^n$ and $d_f\colon \mathbf{B}^n \to \mathbf{R}_+$, $d_f(x) = d(f(x), \partial G)$ satisfies the Harnack inequality (see 12.5 and 3.31).

(3) $f\colon (\mathbf{B}^n, \rho) \to (\mathbf{R}^n, |\ |)$ is uniformly continuous (see also 16.12).

The above sufficient condition 13.7(1) for a qr mapping to be normal may be much refined. As the following important theorem of S. Rickman [RI10] shows it suffices to assume that $\operatorname{card}(\overline{\mathbf{R}}^n \setminus f\mathbf{B}^n)$ exceeds a sufficiently large finite number $p(n, K)$ depending only on n and K. The next result is a qr variant of the Schottky theorem, which has a fundamental role in classical complex analysis [T, p. 268], [A3, p. 19], [NE, p. 62]. Some applications of this result are given in [VU14].

13.8. Theorem ([RI10]). *For $n \geq 3$, $K \geq 1$ there exists $p = p(n, K)$ such that every K-qm mapping $f\colon \mathbf{B}^n \to \overline{\mathbf{R}}^n \setminus \{a_1, \ldots, a_p\}$, where $a_i \neq a_j$ for $i \neq j$, is normal. Moreover, if $\infty \notin f\mathbf{B}^n$, then*

$$\log^+ |f(x)| \leq C_0 \left(-\log s_0 + \log^+ |f(0)|\right) (1 - |x|)^{-C}$$

where $\log^+ t = \log \max\{1, t\}$, $s_0 = \frac{1}{4} \min\{\, q(a_i, a_j) : 1 \leq i, j \leq p, i \neq j \,\}$, and C and C_0 are constants depending only on n, K, and s_0.

We are now going to prove that every normal qr mapping satisfies a growth condition similar to the one in 13.8 and that the constant C can be chosen to depend only on the dimension n and the maximal dilatation K. The proof of such a growth inequality can be based on the Harnack inequality.

13.9. Remark. Let $u\colon \mathbf{H}^2 \to \mathbf{R}_+$ be a harmonic function. By a well-known property of positive harmonic functions u satisfies the Harnack inequality (4.11) with $C_s \leq (\frac{1+s}{1-s})^2$ for each $s \in (0,1)$ (see e.g. [GT, p. 28]). If $f\colon G \to \mathbf{R}^n$ is K-qr, $E = \{z \in G : |f(z)| \leq 1\}$, and $G \setminus E \neq \emptyset$, then $\log |f|$ satisfies (4.11) in each component of $G \setminus E$ with a constant C_s depending only on n, K, and s (see [SE], [MOS], [R12, pp. 232–239], and [GR]).

By virtue of Theorem 13.6 a qr mapping satisfying the hypothesis of the following theorem is normal.

13.10. Theorem. *Let* $f\colon \mathbf{B}^n \to \mathbf{R}^n$ *be a qr mapping with* $0 \in E = \{ z \in \mathbf{B}^n : |f(z)| \leq 1 \}$ *and suppose that there exists a positive number* T *such that* $\rho(x, E) \leq T$ *implies* $|f(x)| \leq e$ *. Then there are positive numbers* β *and* γ*, of which* γ *depends only on* n *and* $K(f)$ *and* β *also on* T*, such that*

$$|f(x)| \leq \exp\left(\beta\left(\frac{1+|x|}{1-|x|}\right)^{\gamma} + 1\right)$$

for all $x \in \mathbf{B}^n$ *.*

Proof. We may assume that $\mathbf{B}^n \setminus E \neq \emptyset$, since there is nothing to prove if $E = \mathbf{B}^n$. For $x \in \mathbf{B}^n \setminus E$ let $u(x) = \log|f(x)|$. We wish to find an upper bound for $|f(z)|$ when z is a fixed prescribed point.

Case 1. $z \in \mathbf{B}^n \setminus E$ and $u(z) > 1$. Now $\rho(z, E) = M > T$. Fix $z_1 \in E$ such that $\rho(z, z_1) = M$. Clearly $u(x) > 0$ for $x \in D(z, M)$ and it follows from 13.9 that u satisfies the Harnack inequality (4.11) in $D(z, M)$ with a constant C_s depending only on n, $K(f)$, and $s \in (0,1)$. Denote

$$F = \{ y \in \mathbf{B}^n : u(y) \leq 1 \}, \quad F_1 = F \cap J[z, z_1].$$

Select $z_2 \in F_1$ with $\rho(z, z_2) = \rho(z, F_1)$. It follows from the hypothesis of the theorem that $\rho(z_2, z_1) \geq T$, while $z_2 \in J[z, z_1]$, $0 \in E$, implies that (see (2.17))

$$\rho(z, z_2) \leq \rho(z, z_1) - T \leq \rho(z, 0) = \log \frac{1+|z|}{1-|z|} \tag{13.11}$$

Denote by $\tilde{\rho}$ the hyperbolic metric of $D(z, M)$ (see 4.25). We are next going to apply 4.12 to $u|D(z, M)$ and to the points z and z_2. For this purpose we need an upper bound for $\tilde{\rho}(z, z_2)$. To find such a bound, map z to 0 by T_z (see 1.34). In order to avoid notational ambiguity denote $h = T_z$. From the conformal invariance of ρ it follows that h maps $D(z, t)$ onto $D(0, t)$, for each $t > 0$. Hence by (2.24) and (2.25) we see that

$$\begin{aligned} h(z_2) &\in S^{n-1}\left(\operatorname{th} \tfrac{1}{2}\rho(z, z_2)\right), \\ hD(z, M) &= B^n\left(\operatorname{th} \tfrac{1}{2}\rho(z, z_1)\right). \end{aligned}$$

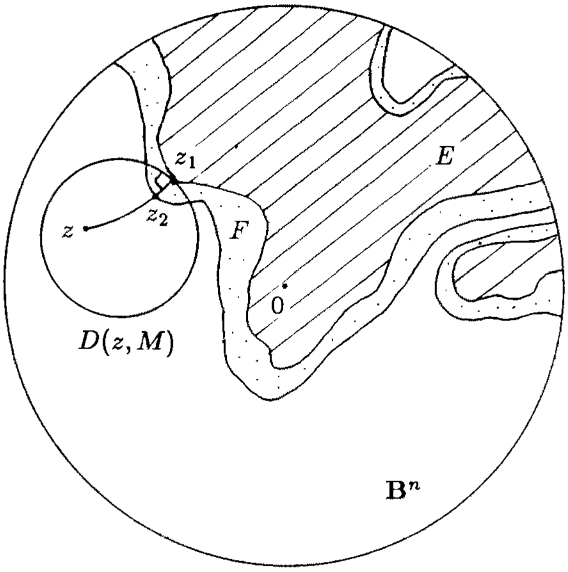


Diagram 13.1. The proof of Theorem 13.10.

In view of the conformal invariance of $\tilde{\rho}$ it follows that (see (2.17) and 4.25)

$$\begin{aligned}\tilde{\rho}_{D(z,M)}(z,z_2) &= \tilde{\rho}_{D(0,M)}(0,h(z_2)) \\ &= \log\frac{1+r}{1-r}\,; \quad r = \frac{\operatorname{th}\frac{1}{2}\rho(z,z_2)}{\operatorname{th}\frac{1}{2}\rho(z,z_1)}\,.\end{aligned} \tag{13.12}$$

Denote $\varphi = \frac{1}{2}\rho(z,z_1)$, $\tau = \frac{1}{2}\rho(z,z_2)$. Because $\varphi - \tau \geq \frac{1}{2}T$ we obtain by (13.11), (13.12), and 2.29(1) the inequalities

$$\begin{aligned}\frac{1+r}{1-r} &= \frac{\operatorname{th}\varphi + \operatorname{th}\tau}{\operatorname{th}\varphi - \operatorname{th}\tau} = \frac{\operatorname{th}(\varphi+\tau)}{\operatorname{th}(\varphi-\tau)}\cdot\frac{1+\operatorname{th}\varphi\operatorname{th}\tau}{1-\operatorname{th}\varphi\operatorname{th}\tau} \\ &< \frac{1}{\operatorname{th}(\varphi-\tau)}\cdot\frac{1+\operatorname{th}\varphi\operatorname{th}\tau}{1-\operatorname{th}^2\varphi} < \frac{1}{\operatorname{th}\frac{1}{2}T}\cdot\frac{1}{1-\operatorname{th}\varphi} < \frac{e^{2\varphi}}{\operatorname{th}\frac{1}{2}T}\,.\end{aligned} \tag{13.13}$$

By 2.29(3), (13.12), (13.13), and (13.11) we get

$$\tilde{\rho}(z,z_2) < \rho(z,z_1) + 2\operatorname{arth} e^{-T}\,. \tag{13.14}$$

Because $0 \in E$ it follows from the choice of z_1 that

$$\rho(z,z_1) \leq \rho(z,0) = \log\frac{1+|z|}{1-|z|}\,. \tag{13.15}$$

Since u satisfies the Harnack inequality, (13.14) and (13.15) together with 4.12 imply that

$$\text{(13.16)}\qquad \begin{cases} u(z) \le C_s\, u(z_2)\exp\left[\gamma\tilde{\rho}(z,z_2)\right] \le \beta\left(\dfrac{1+|z|}{1-|z|}\right)^{\gamma} ; \\ \beta = C_s\left(\dfrac{1+e^{-T}}{1-e^{-T}}\right)^{\gamma} \ , \quad \gamma = (\log C_s)\Big/\log\dfrac{1+s}{1-s} \ . \end{cases}$$

Note that $u(z_2) = 1$ by the choice of z_2. In Case 1 the desired inequality follows from (13.16).

Case 2. $z \in \mathbf{B}^n \setminus E$ and $u(z) \le 1$ or $z \in E$. In this case $|f(z)| \le e$ and hence the assertion is trivial. ❐

We shall next give an alternative proof for 13.10 in the particular case of bounded mappings, since in this case the proof is very short.

13.17. Corollary. *Let $f\colon \mathbf{B}^n \to \mathbf{R}^n$ be a non-constant qr mapping with $f\mathbf{B}^n \subset \mathbf{B}^n$, $y_0 \in \mathbf{B}^n \setminus f\mathbf{B}^n$. Then*

$$\text{(1)}\qquad |f(x) - y_0| \ge 2\exp\left(-A\left(\frac{1+|x|}{1-|x|}\right)^{\gamma}\right)$$

holds for $x \in \mathbf{B}^n$ where $A > 0$ and $\gamma = (\log C_s)/\log\frac{1+s}{1-s}$. Moreover,

$$\text{(2)}\qquad 1 - |f(x)| \ge a\left(\frac{1-|x|}{1+|x|}\right)^{\delta}$$

where $\delta = K_I(f)$ and $a > 0$. Furthermore, if $n = 2$ and $f(0) = 0$ then

$$\text{(3)}\qquad \frac{1-|f(x)|}{1+|f(x)|} \ge 4^{1-\delta}\left(\frac{1-|x|}{1+|x|}\right)^{\delta}$$

for all $x \in \mathbf{B}^n$ where $\delta = K_I(f)$.

Proof. (1) Define $v\colon \mathbf{B}^n \to \mathbf{R}_+$ by $v(x) = -\log\left(\frac{1}{2}|f(x) - y_0|\right)$ for $x \in \mathbf{B}^n$. It follows from 13.9 that the function v satisfies the Harnack inequality (4.11) in $\mathbf{B}^n$ with a constant C_s. Now 4.12 yields

$$v(x) \le v(0)C_s\exp(\gamma\,\rho(0,x))$$

where $\gamma = (\log C_s)/\log\frac{1+s}{1-s}$. The desired bound with $A = v(0)C_s$ follows.

(2) This was proved in Exercise 11.9 with $a = 2^{-2K_I(f)}(1 - |f(0)|)$.

(3) By 11.3, 5.61(3), and 7.47(2)

$$\frac{1+|f(x)|}{1-|f(x)|} \le \frac{1+\varphi_{\delta,2}(|x|)}{1-\varphi_{\delta,2}(|x|)} = 1\Big/\varphi_{1/\delta,2}\left(\frac{1-|x|}{1+|x|}\right)$$
$$\le 4^{\delta-1}\left(\frac{1+|x|}{1-|x|}\right)^{\delta}$$

which yields the desired inequality. ❐

The above results 13.10 and 13.17 depend on a parameter s which can be chosen arbitrarily in $(0,1)$.

13.18. Remark. The exponential function $g\colon \mathbf{B}^2 \to \mathbf{B}^2 \backslash \{0\}$, $g(z) = \exp\left(\frac{z+1}{z-1}\right)$, $z \in \mathbf{B}^2$, shows that an exponential rate of decrease in 13.17(1) can be attained, even by analytic functions. Recall that it was shown in 11.4 that g is *not* uniformly continuous as a map of $(\mathbf{B}^2, \rho)$ into (G, k_G), $G = \mathbf{B}^2 \setminus \{0\}$.

13.19. Exercise. Let G be a proper subdomain of $\mathbf{R}^n$ and let $f\colon \mathbf{B}^n \to \mathbf{R}^n$ be a qr mapping such that $f\colon (\mathbf{B}^n, \rho) \to (G, k_G)$ is uniformly continuous. Let $d_f(x) = d(f(x), \partial G)$. Show that $d_f(x)$ has a lower bound in terms of $d_f(0)$, $\rho(0,x)$, n, and $K_I(f)$, similar to that in 13.17(2). [Hint: Observe that d_f satisfies the Harnack condition (see 12.5 and 12.16). Next apply 4.12.]

We shall next give some corollaries to 13.10. The first one is a Picard theorem for qr mappings. For the statement of this result we call a qm mapping $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}^n$ of the entire space $\mathbf{R}^n$ *normal* if there exists a function $\omega_f : (0,\infty) \to (0,\infty)$ such that $\omega_f(t) \to 0$ as $t \to 0$ and

$$q\big(f(x), f(y)\big) \le \omega_f\big(\rho_R(x,y)\big)$$

for all $R > 0$ and $x, y \in B^n(R)$ where ρ_R is the hyperbolic metric of $B^n(R)$ (cf. 4.25). Equivalently, $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}^n$ is termed normal if

$$q\big(f_k(x), f_k(y)\big) \le \omega_f\big(\rho(x,y)\big)$$

for all $x, y \in \mathbf{B}^n$ and all $k > 1$ where $f_k(z) = f(kz)$, $z \in \mathbf{B}^n$.

13.20. Theorem (Picard's theorem for qr mappings). *A normal qr mapping $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is a constant. In particular, if* $\operatorname{card}\big(\overline{\mathbf{R}}^n \setminus f\mathbf{R}^n\big)$ *is at least the number p of 13.8, then f is a constant.*

Proof. Without loss of generality we may assume that $|f(0)| \le 1$. Fix $z \in \mathbf{R}^n \setminus \mathbf{B}^n$. Applying 13.10 to $f|B^n(2|z|)$ we see that

$$|f(z)| \le \exp(3^\gamma \beta + 1)$$

where γ depends only on n and $K(f)$ and β also on ω_f. This inequality holds for all $z \in \mathbf{R}^n \setminus \mathbf{B}^n$ and hence, by the maximum principle, for all $z \in \mathbf{R}^n$. Thus f is bounded, in contradiction to Theorem 11.15. The second part follows from the Schottky theorem for qr maps, Theorem 13.8. ❐

By a deep recent result of Rickman [RI11] the number q in 13.20 tends to ∞ as $K \to \infty$ if $n = 3$.

13.21. Lemma. *Let* $f\colon \mathbf{B}^n \to \overline{\mathbf{R}}^n$ *be a normal qm mapping, let* (b_k) *be a sequence in* $\mathbf{B}^n$ *with* $b_k \to b \in \partial \mathbf{B}^n$, *let* $f(b_k) \to y \in \overline{\mathbf{R}}^n \setminus f\mathbf{B}^n$ *as* $k \to \infty$, *and let* $M > 0$. *Then* $f(x) \to y$ *as* $x \to b$ *and* $x \in E = \bigcup D(b_k, M)$.

Proof. By performing a preliminary sense–preserving Möbius transformation if necessary, we may assume that $y = \infty$. Assume now that the result is false. Then there exists a sequence (a_k) in E with $a_k \to b$ as $k \to \infty$ and $|f(a_k)| < A < \infty$ for all $k = 1, 2, \ldots$ and some A. Let $g_k \in \mathcal{M}(\mathbf{B}^n)$ with $g_k(0) = a_k$ (see 1.34). Then $f \circ g_k$ is normal in the sense of 13.1, $\omega_{f \circ g_k} = \omega_f$ in view of 13.2(1), and $|(f \circ g_k)(0)|/A \le 1$. By passing to a subsequence and relabeling if necessary we may assume that $\rho(a_k, b_k) < M$ for all K. It follows from 13.10 that

$$|f(b_k)|/A = |f(g_k(g_k^{-1}(b_k)))|/A \le C < \infty$$

for all k where C does not depend on k. This inequality yields a contradiction, since $f(b_k) \to \infty$ as $k \to \infty$. Hence the antithesis is false and the result is proved. □

In the case of normal meromorphic functions of the unit disk in $\mathbf{C}$, Lemma 13.21 can be deduced also from Hurwitz' theorem. An alternative proof of the n–dimensional result 13.21 can be based on the notion of the local topological index and on normal families (see [VU3, 6.3]).

The hypotheses of Lemma 13.21 can be much weakened. This appears from T. Kuusalo's recent result [K2], which shows that Hurwitz' theorem (and hence also 13.21) holds for discrete open normal maps. Note that quasiregularity is not needed here. For a similar result see G. T. Whyburn [WH1].

13.22. Remarks. The hypothesis $y \in \overline{\mathbf{R}}^n \setminus f\mathbf{B}^n$ in 13.21 can be replaced by the slightly weaker requirement that $N(y, f, \mathbf{B}^n) < \infty$. The proof of such an extended version of 13.21 is left as an exercise for the reader. We now give an example to show that the hypothesis $y \in \overline{\mathbf{R}}^n \setminus f\mathbf{B}^n$ cannot be entirely dropped. For this purpose we consider the bounded analytic function $g\colon \mathbf{B}^2 \to \mathbf{B}^2 \setminus \{0\}$, $g(z) = \exp(\frac{z+1}{z-1})$. Fix $\alpha \in \mathbf{B}^2 \setminus \{0\}$ and choose a sequence (b_k) in $\mathbf{B}^2$ with $b_k \to e_1$ and $g(b_k) = \alpha$ for all $k = 1, 2, \ldots$. By studying the properties of g we see that $g(z) \not\to \alpha$ as $z \to e_1$ and

$z \in \bigcup D(b_k, 1)$, i.e. the conclusion of 13.21 fails for this function g if $\alpha \in \mathbf{B}^2 \setminus \{0\}$. Therefore the assumption $y \in \partial f \mathbf{B}^n$ cannot be dropped from 13.21.

13.23. Exercise. In the particular case when $\partial f \mathbf{B}^n$ is a non–degenerate continuum, one can deduce 13.21 by applying the K_I–inequality. Give the details. [Hint: Apply 11.5(1).]

13.24. Notes. This section is taken from [VU10]. For 13.21 see [BS]. An account of Schottky's theorem can be found in [BU, Ch. XII]. In the particular case of analytic functions, 13.17 can be found in [HM].

Chapter IV
BOUNDARY BEHAVIOR

In the present chapter we shall study the behavior of qc and qr mappings near the boundary of the domain of definition. There is a fundamental difference in the study of these two classes of mappings: in the case of qr mappings the maximal multiplicity of the mapping may be infinite even in every neighborhood of a given boundary point. Consequently, one cannot apply the important K_O-inequality to the qr case in the same way as to the qc case because of the presence of a multiplicity factor which may be infinite. Many differences in the theories of qc and qr mappings are more or less directly connected with this fact.

The fundamental problem which we are going to study in this chapter is the following.

Problem. Let $f\colon \mathbf{B}^n \to \mathbf{R}^n$, $n \geq 2$, be a qc or qr mapping, $b \in \partial\mathbf{B}^n$, and let $E \subset \mathbf{B}^n$ be a set with $b \in \overline{E}$ and $f(x) \to \alpha$ as $x \to b$, $x \in E$. Under which conditions on f and E is α in fact an angular limit of f at b?

By Lindelöf's well-known result this is the case if f is a bounded analytic function and E is a curve terminating at b. We shall show by exploiting the K_O-inequality that if E is thick enough at b in the sense of n-capacity and if f is qc, then f has an angular limit α at b. We give an example to show that the thickness condition is in a sense best possible.

We shall also discuss the case where f is qr. Under the additional assumption that f be Dirichlet-finite we shall extend the above result about qc maps to the case of qr mappings. We shall investigate also some other properties of Dirichlet-finite qr mappings.

14. Some properties of quasiconformal mappings

We shall introduce some notation and terminology, useful in the discussion of boundary behavior, and then prove some results about qc mappings. The presentation is aimed to be self–contained also in this chapter. Those readers who wish to find some background, motivation, or further results on boundary behavior of analytic functions are referred to [CL], [NO], [LOH], or [PO]. Of these, [LOH] contains an extensive bibliography. For the boundary behavior of qc mappings see [N1] and [N3].

14.1. Definition. Let $f\colon \mathbf{H}^n \to \mathbf{R}^n$ be continuous. The mapping f is said to have

(1) a *sequential limit* $\alpha \in \overline{\mathbf{R}}^n$ at 0 if there exists a sequence (b_k) in $\mathbf{H}^n$ with $b_k \to 0$ and $f(b_k) \to \alpha$;

(2) an *asymptotic value* $\alpha \in \overline{\mathbf{R}}^n$ at 0 if there exists a curve $\gamma : [0,1) \to \mathbf{H}^n$, termed an *asymptotic curve*, such that $\gamma(t) \to 0$ and $f(\gamma(t)) \to \alpha$ as $t \to 1$;

(3) an *angular limit* $\alpha \in \mathbf{R}^n$ at 0 if, for each $\varphi \in (0, \frac{1}{2}\pi)$, $f(x)$ approaches α as x tends to 0 in $C(\varphi)$ (for the definition of the cone $C(\varphi)$ see the definition preceding Theorem 11.17);

(4) a *limit* $\alpha \in \overline{\mathbf{R}}^n$ *at* 0 *through a set* E, if $0 \in \overline{E} \subset \mathbf{H}^n \cup \{0\}$ and $f(x) \to \alpha$ as $x \to 0$ and $x \in E$.

The set $C(f,b)$ of all sequential limits of f at a boundary point b is termed the *cluster set* of f at b. If $A \subset \partial\mathbf{H}^n$ is non–empty, we denote $C(f,A) = \bigcup_{b\in A} C(f,b)$.

In the literature an angular limit is sometimes called a non–tangential or (if $n \geq 3$) a conical limit. It is clear that $C(f,b)$ is always a compact non–empty subset of $\overline{f\mathbf{H}^n}$. From the well–known fact that one–to–one mappings preserve open sets, it follows that $C(f,b) \subset \partial f\mathbf{H}^n$ for one–to–one mappings f (see also 9.12).

14.2. Remarks. (1) The cluster set of $f\colon \mathbf{H}^n \to \overline{\mathbf{R}}^n$ at $b \in \partial\mathbf{H}^n$ can alternatively be defined as

$$C(f,b) = \bigcap_U \overline{f(U \cap \mathbf{H}^n)}$$

where U runs through all neighborhoods of b. From this definition it follows that $C(f,b)$ is connected whenever f is a continuous mapping of $\mathbf{H}^n$. (More generally, $C(f,b)$ is connected if f is defined on a domain G which is locally connected at $b \in \partial G$ [CL, p. 3].)

(2) For $x \in \overline{\mathbf{R}}^n$, $\epsilon > 0$, let $E_\epsilon^x = \{ z \in G : q(f(z),x) < \epsilon \}$. A second equivalent definition of $C(f,b)$ is

$$C(f,b) = \{ z \in \overline{\mathbf{R}}^n : b \in \overline{E}_\epsilon^z \text{ for all } \epsilon > 0 \}.$$

(3) It is clear that $q\big(f(x), C(f,b)\big) \to 0$ as $q(x,b) \to 0$, $x \in \mathbf{H}^n$. In particular, $C(f,b) = \{b'\}$ iff f has a limit b' at b.

14.3. Examples. (1) By the Riemann mapping theorem there exists a conformal mapping $f\colon \mathbf{H}^2 \to D$, where D is as pictured.

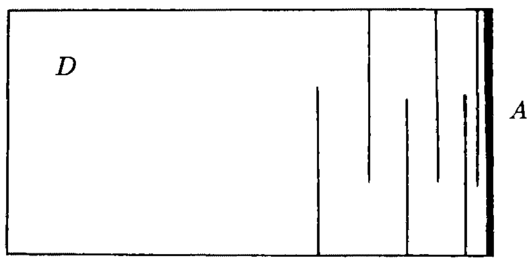


Diagram 14.1.

It follows from the theory of prime ends (cf. [CL], [D], [PO], [NO]) that the segment A corresponds to a single point $b \in \partial \mathbf{H}^2$ under f, i.e. $C(f,b) = A$. In this case f has no asymptotic value, hence no angular limit at b.

(2) A conformal mapping $f\colon \mathbf{H}^2 \to G$ with $C(f,0) = B$, where G is the domain and B the segment pictured, has an angular limit but not an ordinary limit at 0 (cf. [CL], [O]).

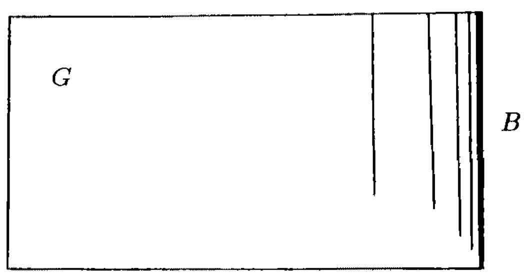


Diagram 14.2.

(3) It seems difficult to give an explicit expression for the conformal mapping f in (2). We now exhibit an example of a real–valued function with properties somewhat analogous to those of (2), i.e. this function will have an angular limit but not a limit at a boundary point. Let $u\colon \mathbf{H}^2 \to (0,\infty)$ be defined by $u(x,y) = (x^2+y^2)/y$ for $(x,y) \in \mathbf{H}^2$. Then u has no limit at 0, in fact $C(f,0) = [0,\infty]$, but it does have an angular limit 0 at 0 and an asymptotic value $c > 0$ through the circular arc $\{(x,y) \in \mathbf{H}^2 : x^2 + (y - \frac{1}{2}c)^2 = \frac{1}{4}c^2, x > 0\}$.

(4) The harmonic function $u\colon \mathbf{H}^2 \to (0,\pi)$, $u(x,y) = \arctan(y/x)$, $(x,y) \in \mathbf{H}^2$, has a constant value on each ray in $\mathbf{H}^2$ emanating from the origin.

(5) The function $v\colon \mathbf{H}^2 \to [0,1]$, $v(x,y) = \sin^2(1/\sqrt{x^2+y^2})$ has no asymptotic value at 0, hence no angular limit at 0. The function $v_1\colon \mathbf{H}^2 \to [0,1]$, $v_1(x,y) = v(x,y)y/\sqrt{x^2+y^2}$ has an asymptotic value 0 at 0 but no angular limit at 0.

A conformal mapping of $\mathbf{H}^2$ may fail to have an angular limit at a boundary point $b \in \partial\mathbf{H}^2$ (cf. 14.3(1)). However, the set of all such points of $\partial\mathbf{H}^2$ is very small; it is of capacity zero by Beurling's theorem (see 14.7 below). A bounded analytic function of the unit disc $\mathbf{B}^2$ has an angular limit at each point of $\partial\mathbf{B}^2$ except possibly for a subset of $\partial\mathbf{B}^2$ of linear measure zero (Fatou's theorem [CL, p. 17]).

A set $E \subset \mathbf{H}^n$ is said to be *non–tangential* at 0, if $0 \in \overline{E} \subset \mathbf{H}^n \cup \{0\}$ and $E \subset C(\varphi)$ for some $\varphi \in (0, \frac{1}{2}\pi)$, and *tangential* at 0 if $0 \in \overline{E} \subset \mathbf{H}^n \cup \{0\}$ and $E \not\subset C(\varphi)$ for each $\varphi \in (0, \frac{1}{2}\pi)$.

14.4. Remark. Suppose that a mapping $f\colon \mathbf{H}^n \to \overline{\mathbf{R}}^n$ has an angular limit α at 0. It follows almost immediately from the definition of an angular limit that f approaches α not only through each non–tangential set but also through a set E which is tangential at 0. In fact, by the definition 14.1(3), for each $k = 1,2,\ldots$ there exist r_k, $r_{k+1} \in (0, \frac{1}{2}r_k)$, such that $x \in E_k$, $E_k = C(\frac{\pi k}{2(k+1)}) \cap B^n(r_k)$, implies $q(f(x),\alpha) < 1/k$. Clearly f approaches α through the tangential set $E = \bigcup_{k=1}^{\infty} E_k$. However, the "degree of contact" between E and $\partial\mathbf{H}^n$ depends on f.

In the preceding discussion we have considered real– or vector–valued continuous mappings of $\mathbf{H}^n$. Some of the above definitions have natural counterparts for mappings of an arbitrary domain G in $\mathbf{R}^n$, which we shall use if necessary. Now we are going to consider normal mappings (cf. 13.1). The following lemma should be compared to 13.21.

14.5. Lemma. *Let* $f\colon \mathbf{H}^n \to \overline{\mathbf{R}}^n$ *be normal, let* $b_k \in \mathbf{H}^n$, $b_k \to 0$, *and let* $f(b_k) \to \beta$. *For every* $\epsilon > 0$ *there exist* $M \in (0,\infty)$ *and* $p \geq 1$ *such that*

$$q(f(x),\beta) < \epsilon \quad \text{for} \quad x \in E_M = \bigcup_{k \geq p} D(b_k, M) \,.$$

Moreover, if there exists an angle $\varphi \in (0, \frac{1}{2}\pi)$ *such that* $b_k \in C(\varphi)$ *for all* k , *then* $m_1([0,b_k] \cap E_M) \geq d(\varphi, M) > 0$.

Proof. The first part follows from the definition of a normal function. For the second part note that

$$B^n\big(b_k, b_{kn}(1-e^{-M})\big) \subset D(b_k, M)$$

by (2.11), where b_{kn} is the nth coordinate of b_k . Since $b_k \in C(\varphi)$, we see that $b_{kn} \geq |b_k| \cos\varphi$, and the assertion follows with $d(\varphi, M) = (1-e^{-M})\cos\varphi$. ☐

14.6. Lemma. *If* $f\colon \mathbf{H}^n \to G' = f\mathbf{H}^n$ *is a qc mapping then the following assertions hold.*

(1) *The set* $\partial G' = C(f, \partial\mathbf{H}^n)$ *is a non-degenerate continuum.*

(2) *If* $E_j \subset \mathbf{H}^n$, $0 \in \overline{E}_j$, *and* $\overline{fE_1} \cap \overline{fE_2} = \emptyset$, *then* $\mathsf{M}(\Delta(E_1, E_2; \mathbf{H}^n)) < \infty$.

Proof. (1) Because one–to–one continuous maps (and their inverses) preserve open sets the set–theoretic equality $\partial G' = C(f, \partial\mathbf{H}^n)$ is clear. It follows easily from 14.2 that

$$C(f, \partial\mathbf{H}^n) = \bigcap_{j=2}^{\infty} \overline{fU_j} \quad ; \quad U_j = \mathbf{H}^n \setminus D(e_n, j) \,.$$

Because f is continuous, the sequence $\{\, \overline{fU_j} : j = 2, 3, \ldots\}$ is a decreasing sequence of connected compact sets of $\overline{\mathbf{R}}^n$ and thus $C(f, \partial\mathbf{H}^n)$ is connected by a well–known topological result. By 8.6(1), (7.31), (2.6), and 10.19

$$0 < 2^{n-1} c_n \log 2 \leq \mu_{\mathbf{H}^n}(e_n, 2e_n) \leq K_O(f)\, \mu_{f\mathbf{H}^n}\big(f(e_n), f(2e_n)\big) \,,$$

and hence $C(f, \partial\mathbf{H}^n)$ contains more than two points; that is, $C(f, \partial\mathbf{H}^n)$ is non–degenerate.

(2) Because f is one–to–one $f\Delta(E_1, E_2; \mathbf{H}^n) = \Delta(fE_1, fE_2; f\mathbf{H}^n)$ and thus by 5.23 or by 6.20 and (10.11)

$$\mathsf{M}(\Delta(E_1, E_2; \mathbf{H}^n)) \leq K_O(f)\, \mathsf{M}(\Delta(\overline{fE_1}, \overline{fE_2}; f\mathbf{H}^n)) < \infty \,. \quad ☐$$

The following result is a generalization of Beurling's theorem on conformal mappings (see [CL, p. 56, Theorem 3.5], [N3]).

14.7. Lemma. *Let* $f\colon \mathbf{B}^n \to G'$ *be qc and*

$$E = \{ b \in \partial \mathbf{B}^n \ : \ f \text{ has no asymptotic value at } b \} .$$

If $F \subset E$ *is compact, then* $\operatorname{cap} F = 0$.

Proof. Assume that $F \subset E$ is compact and $\operatorname{cap} F > 0$. Let $K = \overline{\mathbf{B}}^n(\frac{1}{2})$ and $\Gamma = \Delta(K, F; \mathbf{B}^n)$. Denote by Γ_r the family of all rectifiable paths in Γ and by Γ'_r the family of all rectifiable paths in $f\Gamma_r$. Then by 5.8, 5.20, 6.1(5)

$$\mathsf{M}(\Gamma'_r) = \mathsf{M}(f\Gamma_r) \geq \mathsf{M}(\Gamma_r)/K(f) = \mathsf{M}(\Gamma)/K(f) > 0$$

because $\operatorname{cap} F > 0$. Hence $\Gamma'_r \neq \emptyset$. Thus there exists a rectifiable path $\gamma \in \Gamma_r$ such that $f \circ \gamma$ is rectifiable, i.e. f has a limit through $|\gamma|$. This contradicts the choice of F . ☐

14.8. An open problem. This problem, due to F. W. Gehring, has been studied by P. Caraman [C2]. Let $f\colon \mathbf{B}^n \to G'$ be a qc mapping and $E_{nr} = \{ b \in \partial \mathbf{B}^n : f[\frac{1}{2}b, b) \text{ is non-rectifiable} \}$. For a Borel set $A \subset E_{nr}$ let $\Gamma_A = \{ [\frac{1}{2}b, b) \ : \ b \in A \}$. Then every path in $f\Gamma_A$ is non-rectifiable and hence $\mathsf{M}(f\Gamma_A) = 0$ by 5.8. It follows from (5.13) that also

$$\mathsf{M}(\Gamma_A) = m_{n-1}(A)(\log 2)^{1-n} = 0$$

and hence $m_{n-1}(A) = 0$ whenever $A \subset E_{nr}$ is a Borel set. *Problem:* Is it true that $\operatorname{cap} F = 0$ for every compact subset F of E_{nr} ?

For the following chapters we shall need a convenient criterion for the thickness of a set $E \subset \overline{\mathbf{R}}^n$ at a point $x \in \mathbf{R}^n$. The *lower* and *upper capacity densities* of E at x are defined by (cf. [VU2], [VU3])

$$\text{(14.9)} \qquad \begin{aligned} \operatorname{cap} \underline{\operatorname{dens}}(E, x) &= \liminf_{r \to 0} M(E, r, x) \ , \\ \operatorname{cap} \overline{\operatorname{dens}}(E, x) &= \limsup_{r \to 0} M(E, r, x) \ , \end{aligned}$$

where $M(E, r, x)$ is as in (6.2). Set $A_x = \{ r > 0 \ : \ S^{n-1}(x, r) \cap E \neq \emptyset \}$ for $x \in \mathbf{R}^n$. If A_x is measurable we define the *lower* and *upper radial densities* of E at x , respectively, by

$$\text{(14.10)} \qquad \begin{aligned} \operatorname{rad} \underline{\operatorname{dens}}(E, x) &= \liminf_{r \to 0} \frac{m_1\big((0, r) \cap A_x\big)}{r} \ , \\ \operatorname{rad} \overline{\operatorname{dens}}(E, x) &= \limsup_{r \to 0} \frac{m_1\big((0, r) \cap A_x\big)}{r} \ , \end{aligned}$$

where m_1 is Lebesgue measure on $\mathbf{R}$. It is not difficult to see that A_x is measurable for every $x \in \mathbf{R}^n$ if E is open or closed.

14.11. Lemma. *If* E *is a compact subset of* $\mathbf{R}^n$ *with* $\operatorname{rad}\underline{\operatorname{dens}}(E,0) \geq \delta > 0$, *then* $\operatorname{cap}\underline{\operatorname{dens}}(E,0) \geq c(n,\delta) > 0$, *where* $c(n,\delta)$ *depends only on* n *and* δ.

The proof of this lemma is a straightforward application of spherical symmetrization. The details can be found in [VU3]. It is clear that a similar result holds also for upper densities.

14.12. Examples. (1) Let $S_k = S^{n-1}(2^{-k}) \cap \{x : x_n \geq 0\}$, $k = 1,2,\ldots$, and let $E = \{0\} \cup (\bigcup S_k)$. It follows from 5.34 that $\operatorname{cap}\underline{\operatorname{dens}}(E,0) > 0$, while clearly $\operatorname{rad}\overline{\operatorname{dens}}(E,0) = 0 = \operatorname{rad}\underline{\operatorname{dens}}(E,0)$.

(2) There exists a compact set E of zero Hausdorff dimension such that $\operatorname{cap}\underline{\operatorname{dens}}(E,0) > 0$. By a well-known result, see 7.15(1), there exists a compact Cantor-type set $E_1 \subset B^n(2) \setminus \overline{\mathbf{B}}^n$ of positive capacity and zero Hausdorff dimension. Exploiting this fact we construct a set E with the desired property. Let $h\colon \mathbf{R}^n \to \mathbf{R}^n$ be the mapping $h(x) = \frac{1}{2}x$, $x \in \mathbf{R}^n$, and denote $E_{k+1} = hE_k$. The set $E = \{0\} \cup (\bigcup E_k)$ is compact and of zero Hausdorff dimension. Since $\operatorname{cap} E_1 > 0$, also $M(E_1,4,0) = \delta > 0$ (see 6.1(5)). Hence also $\operatorname{cap}\underline{\operatorname{dens}}(E,0) \geq \delta$.

14.13. Remarks. (1) It is possible to construct a compact Cantor set E on the positive x_1-axis such that $m_1(E) = 0$, $\operatorname{cap}\underline{\operatorname{dens}}(E,0) > 0$, and $\operatorname{rad}\overline{\operatorname{dens}}(E,0) = 0$. Therefore, in some cases there are no positive lower bounds for the capacity density in terms of the radial density. Sometimes one can exploit other lower bounds for the capacity densities, see [M4].

(2) The condition $\operatorname{cap}\underline{\operatorname{dens}}(E,0) \geq \delta > 0$ is sometimes used in the following way. First fix $r_0 > 0$ such that $M(E,r,0) \geq \frac{3}{4}\delta$ for $r \in (0,r_0)$. Next choose $\lambda = \lambda(n,\delta) > 2$ such that $\omega_{n-1}(\log 2\lambda)^{1-n} = \frac{1}{4}\delta$. Then

$$M(\overline{B}^n(r/\lambda),\, r,\, 0) \leq \omega_{n-1}(\log 2\lambda)^{1-n} = \tfrac{1}{4}\delta$$

for all $r \in (0,r_0)$. Let $E_1 = E \cap \big(\overline{B}^n(r) \setminus B^n(r/\lambda)\big)$ and $E_2 = E \cap \overline{B}^n(r/\lambda)$. Further, by 5.9,

$$M(E,r,0) \leq M(E_1,r,0) + M(E_2,r,0)$$

and hence $M(E_1,r,0) \geq \frac{1}{2}\delta$.

The next lemma gives a condition for a curve family to have infinite modulus generalizing 5.33 (cf. [VU2]).

14.14. Lemma. *If* $\operatorname{cap}\underline{\operatorname{dens}}(E_1,0) = \delta_1 > 0$ *and* $\operatorname{cap}\overline{\operatorname{dens}}(E_2,0) = \delta_2 > 0$, *then* $M(\Delta(E_1,E_2)) = \infty$.

Proof. Fix $r_0 \in (0,1)$ such that $M(E_1,r,0) \geq \frac{3}{4}\delta$ for all $r \in (0,r_0)$ and let $\lambda_1 = \lambda_1(n,\delta_1)$ be the number in 14.13(2). Fix a sequence $r_1 > r_2 > \dots$ such that $r_1 \in (0,r_0)$ and $M(E_2,r_j,0) \geq \frac{3}{4}\delta_2$ for $j = 1,2,\dots$ and let $\lambda_2 = \lambda_2(n,\delta_2)$ be as in 14.13(2). Denote $\lambda = \max\{\lambda_1,\lambda_2\}$. Then

$$\omega_{n-1}(\log 2\lambda)^{1-n} = \tfrac{1}{4}\min\{\,\delta_1,\,\delta_2\,\}\,. \tag{14.15}$$

Fix j and denote $F_i = E_i \cap \left(\overline{B}^n(r_j) \setminus B^n(r_j/\lambda)\right)$, $i = 1,2$, $F_3 = S^{n-1}(2r_j)$. Applying 5.41 to the triple F_1,F_2,F_3 we obtain as in 14.13(2)

$$\mathsf{M}(\Delta(F_1,F_2)) \geq 2\,d(n)\min\{\,\delta_1,\,\delta_2\,\}$$

where $d(n) = 2^{-2}3^{-n}\min\{1, c_n(\log 2)^n/\omega_{n-1}\}$. Next we are going to select a positive number $\mu = \mu(n,\delta_1,\delta_2)$ such that

$$\mathsf{M}(\Delta(F_1,F_2;R_j^\mu)) \geq d(n)\min\{\,\delta_1,\,\delta_2\,\} \tag{14.16}$$

where $R_j^\mu = B^n(2\mu r_j) \setminus \overline{B}^n(r_j/(2\lambda\mu))$. Since $F_1,F_2 \subset \overline{B}^n(r_j) \setminus B^n(r_j/\lambda)$ it follows from 5.9 and (5.14) that it suffices to choose μ so that

$$2\,\omega_{n-1}(\log 2\mu)^{1-n} \leq d(n)\min\{\,\delta_1,\,\delta_2\,\}\,. \tag{14.17}$$

We shall next find an upper bound for μ in terms of λ and n. It follows from (14.15) that

$$\omega_{n-1}\left(\log(2\lambda)^p\right)^{1-n} \leq \tfrac{1}{4}\,p^{1-n}\min\{\,\delta_1,\,\delta_2\,\}\,.$$

Hence (14.17) is fulfilled as soon as $2\mu \geq (2\lambda)^p$ and $\frac{1}{2}p^{1-n} \leq d(n)$. Let $p_0 \geq 1$ be the least integer satisfying this last inequality and set $\mu = (2\lambda)^{p_0}$. With this choice of μ (14.17) holds. By passing to a subsequence of (r_j), if necessary, we may assume that the rings R_j^μ are separate and that (14.16) holds for all j. It follows from 5.4 and (14.16) that

$$\mathsf{M}(\Delta(E_1,E_2)) \geq \sum_{j=1}^{\infty}\mathsf{M}(\Delta(E_1,E_2;R_j^\mu)) = \infty\,. \quad \square$$

14.18. Example. There exist sets E and F with $\operatorname{cap}\overline{\operatorname{dens}}(E,0) > 0$, $\operatorname{cap}\overline{\operatorname{dens}}(F,0) > 0$ and $\mathsf{M}(\Delta(E,F)) < 1$: Let $r_0 = 1$ and choose $r_{j+1} \in (0, \frac{1}{2}r_j)$ such that

$$2\sum_{j=1}^{\infty} \omega_{n-1}\Big(\log \frac{r_j}{r_{j+1}}\Big)^{1-n} < 1 .$$

Set $E = \bigcup_{j=1}^{\infty} S^{n-1}(r_{2j-1})$ and $F = \bigcup_{j=1}^{\infty} S^{n-1}(r_{2j})$. By 5.9 and (5.14)

$$\mathsf{M}(\Delta(E,F)) \le \sum_{j=1}^{\infty} \mathsf{M}(\Delta(E,F_j))$$
$$\le \sum_{j=1}^{\infty} \omega_{n-1}\Big[\Big(\log \frac{r_j}{r_{j+1}}\Big)^{1-n} + \Big(\log \frac{r_{j-1}}{r_j}\Big)^{1-n}\Big] < 1 .$$

14.19. Exercise. Applying 14.6(2) and 5.33 show that a qc mapping of $\mathbf{H}^n$ cannot have two distinct asymptotic values at a point $b \in \partial\mathbf{H}^n$. Applying 14.6(2) and 14.14 one can generalize this observation as follows. If a qc mapping of $\mathbf{H}^n$ has a limit α_j through a set E_j at 0, $j = 1,2$, and if $\alpha_1 \neq \alpha_2$, then it is not possible that both $\operatorname{cap}\underline{\operatorname{dens}}(E_1,0) > 0$ and $\operatorname{cap}\overline{\operatorname{dens}}(E_2,0) > 0$ hold.

14.20. Exercise. Let $E \subset \mathbf{H}^n$ be non-tangential at 0 and let $f\colon \mathbf{R}^n \to \mathbf{R}^n$ be a K-qc mapping with $f\mathbf{H}^n = \mathbf{H}^n$ and $f(0) = 0$. Show that fE is non-tangential at 0. [Hint: Apply 12.12 to $f|\,\mathbf{R}^n \setminus \{0\}$ and make use of the fact that $f(\partial\mathbf{H}^n) = \partial\mathbf{H}^n$.] See also [MOR2].

15. Lindelöf-type theorems

From a result of E. Lindelöf it follows that a conformal mapping of $\mathbf{B}^2$ having an asymptotic value α at a boundary point b also has an angular limit α at b. A similar result was proved by Gehring [G3, p. 21] in the case of qc mappings in $\mathbf{R}^3$, and the same proof applies to the n-dimensional case. The following result weakens the hypothesis about the existence of an asymptotic value.

15.1. Theorem. *Let $f\colon \mathbf{H}^n \to G'$ be a qc mapping, and let $E \subset \mathbf{H}^n$ be such that $0 \in \overline{E}$ and $\operatorname{cap}\underline{\operatorname{dens}}(E,0) > 0$. If $f(x) \to \alpha$ as $x \to 0$, $x \in E$, then f has an angular limit α at 0.*

Proof. Suppose, on the contrary, that there exist $\varphi \in (0, \frac{1}{2}\pi)$ and a sequence (b_k) in $C(\varphi)$ with $f(b_k) \to \beta$ and $\alpha \neq \beta$. By performing an auxiliary Möbius transformation we may assume that $\alpha, \beta \neq \infty$. Let $3r = |\alpha - \beta|$. As a qc mapping of $\mathbf{H}^n$, f is normal (cf. 13.7(1)) and it follows from 14.5 that there exist numbers $M > 0$ and $r_0 > 0$ such that

$$\text{(15.2)} \qquad \begin{cases} fE_1 \subset B^n(\alpha, r)\,, & E_1 = B^n(r_0) \cap E\,; \\ fE_2 \subset B^n(\beta, r)\,, & E_2 = B^n(r_0) \cap \left(\bigcup D(b_k, M) \right). \end{cases}$$

We denote $\Gamma = \Delta(E_1, E_2; \mathbf{H}^n)$. By (5.14) and (5.2) $\mathsf{M}(f\Gamma) < \infty$. Since $b_k \in C(\varphi)$ it follows from 14.5 that $\operatorname{rad}\overline{\operatorname{dens}}(E_2, 0) > 0$. On the other hand we get by 5.22, 14.11, and 14.14 that

$$\mathsf{M}(\Gamma) \geq \tfrac{1}{2}\,\mathsf{M}\big(\Delta(E_1, E_2; \mathbf{R}^n)\big) = \infty\,.$$

This inequality contradicts (15.2) and $\mathsf{M}(\Gamma) \leq K_O(f)\,\mathsf{M}(f\Gamma)$. ❐

15.3. Remarks. (1) The condition $\operatorname{cap}\underline{\operatorname{dens}}(E, 0) > 0$ in 15.1 cannot be replaced by $\operatorname{cap}\overline{\operatorname{dens}}(E, 0) > 0$. To prove this statement we consider a conformal mapping $f\colon \mathbf{H}^2 \to G'$ having no limits along the y-axis at 0. For the existence of such a mapping the reader is referred to the theory of prime ends (cf. references given in 14.3). Let $C_\perp(f, 0)$ be the cluster set of f at 0 along the y-axis and fix $\alpha \in C_\perp(f, 0)$. By the definition of $C_\perp(f, 0)$ there are numbers $t_k \searrow 0$ with $f(t_k e_2) \to \alpha$. By 13.23 $f(x) \to \alpha$ as $x \to 0$, $x \in \bigcup D(t_k e_2, 1)$, and we see by 14.5 (or more directly, by (2.11)) that

$$\operatorname{rad}\overline{\operatorname{dens}}\big(\bigcup D(t_k e_2, 1), 0\big) > 0\,,$$

and hence also the upper capacity density is positive by the proof of 14.11. The function f has the desired properties, since it fails to have an angular limit at 0.

(2) The main interest in 15.1 lies in the case of a tangential set E. If E is non-tangential at 0 and if $\operatorname{cap}\underline{\operatorname{dens}}(E, 0) > 0$ then, as we shall show in 15.7, E contains a sequence (b_k) with $b_k \to 0$ and $\limsup \rho(b_k, b_{k+1}) < \infty$. From this fact and from 13.21 and from Gehring's result [G3] one gets a simple proof of 15.1 in case E is tangential at 0.

To ensure the measurability required for the definition (14.10) of a radial density we assume in the following theorem that E is either open or closed. This is no restriction of generality, since from the fact that $f(x) \to \alpha$, $x \to 0$, $x \in E$ it follows by elementary properties of continuous mappings that f has a limit α at 0 through an open set F with $E \subset F$ whether E is open or not. A result analogous to 15.4(1) for bounded analytic functions is due to T. Hall [H].

15.4. Corollary. *Let $f\colon \mathbf{H}^n \to G'$ be a qc mapping, let $E \subset \mathbf{H}^n$ be an open or closed set with $0 \in \overline{E}$ and $f(x) \to \alpha$, $x \to 0$, $x \in E$. Then f has an angular limit α at 0 if one of the following conditions is satisfied.*

(1) *E is a curve terminating at 0 or, more generally, E is a set with* $\operatorname{rad} \underline{\operatorname{dens}}(E,0) > 0$.

(2) $E = \{ b_k : k = 1,2,\ldots\}$ *where* $b_k \in \mathbf{H}^n$ *and* $b_k \to 0$, *and* $\limsup \rho(b_k, b_{k+1}) < \infty$.

(3) $\operatorname{cap} \underline{\operatorname{dens}}(E_M, 0) > 0$, *where* $E_M = \bigcup_{x \in E} D(x,M)$ *and* $M \in (0,\infty)$.

Proof. Part (1) follows from 14.11 and 15.1. For the proof of (2) suppose that $\rho(b_k, b_{k+1}) < M$ for $k \geq k_0$. Then the set $E_M = \bigcup_{k \geq k_0} D(b_k, M)$ is connected and f has the same limit α through E_M by 13.21 (or by Exercise 13.23). After this observation, part (2) follows from (1). Part (3) follows from 13.23 and 15.1. ❐

When we compare the above condition 15.4(1) with (3), the following question arises. Suppose (1) holds. Does there exist $M \in (0,\infty)$ and a sequence (b_k) in E with $b_k \to 0$ and $\rho(b_k, b_{k+1}) < M$? The answer is negative, as the following example shows.

15.5. Example. There exists a set $E \subset \mathbf{H}^n$ with $\operatorname{rad} \underline{\operatorname{dens}}(E,0) = 1$ such that

$$\limsup \rho(b_k, b_{k+1}) = \infty \tag{15.6}$$

for every sequence (b_k) in E with $b_k \to 0$. Set $E_k = [2^{-k-1}e_1, 2^{-k}e_1] + t_k e_n$, where $t_k / t_{k+1} = k$, and $E = \bigcup E_k$. Then it follows from (2.11) that

$$\rho\Big(E_k, \bigcup_{j \neq k} E_j \Big) \longrightarrow \infty \quad \text{as} \quad k \longrightarrow \infty .$$

Hence (15.6) is clearly satisfied, and E has the desired properties.

An essential feature of the above example is that the set E is tangential at 0. Indeed, we shall show that such an example is impossible if E is non–tangential.

15.7. Non–tangential sets. Let the set $E \subset \mathbf{H}^n$ be non–tangential at 0 with $\operatorname{cap}\underline{\operatorname{dens}}(E,0) = 2\delta > 0$ and $\varphi \in (0, \frac{1}{2}\pi)$ such that $E \subset C(\varphi)$. Choose $r_0 \in (0,1)$ such that $M(E,r,0) \geq \delta$ for $r \in (0,r_0)$ and $\lambda > 1$ such that

$$\omega_{n-1}(\log 2\lambda)^{1-n} \leq \tfrac{1}{2}\delta \, .$$

Then it follows from Remark 14.13(2) that for each $r \in (0,r_0)$ there exists a point $b_r \in E \cap \overline{R}(r,r/\lambda,0)$ where $R(r,r/\lambda,0) = B^n(r) \setminus \overline{B}^n(r/\lambda)$. Let $r_k = r_0/(2\lambda^k)$ and $b_k = b_{r_k}$. By 4.23 we get

$$\rho(b_k, b_{k+1}) \leq \rho\big(C(\varphi) \cap R(r, r/\lambda^2, 0)\big) = c(\varphi, n, \delta) < \infty \, .$$

This inequality shows that (b_k) is the desired sequence.

We shall next compare the hypotheses of 15.1 with those of 15.4(1).

15.8. Example. If the dimension $n \geq 3$, then there exists a set $E \subset \mathbf{H}^n$ such that $\operatorname{cap}\underline{\operatorname{dens}}(E,0) > 0$ but $\operatorname{rad}\overline{\operatorname{dens}}(E_M,0) = 0$ for all $M > 0$. For simplicity let $n = 3$ and define E by

$$E = \bigcup_{k=1}^{\infty} \{\, (x,y,z) \in \mathbf{H}^3 : x^2 + y^2 = 2^{-2k},\ z = 2^{-k}/k \,\} \, .$$

Fix $M > 0$. Let $E_M = \bigcup_{x \in E} D(x,M)$ and $A = \{\, r > 0 : S^{n-1}(r) \cap E_M \neq \emptyset \,\}$. Clearly $\operatorname{cap}\underline{\operatorname{dens}}(E,0) > 0$ (the dimension $n \geq 3$). By (2.11) the lengths of the components of A have an upper bound $e^M 2^{-k}/k$ and it follows that $\operatorname{rad}\overline{\operatorname{dens}}(E_M,0) = 0$ (for more details, see [VU2, 6.9(3)]). It seems to be an open question whether a set with similar properties can be constructed in $\mathbf{H}^2$, too.

We shall next prove a generalization of the above Lindelöf–type theorem 15.1, which is motivated by a theorem of J. L. Doob [D]. Consider a qc mapping $f \colon \mathbf{H}^n \to G'$ with $0 \in C(f,0)$ (this condition is just a normalization). We want to find a condition, as general as possible, which implies that f has an angular limit 0 at 0. Denote

$$E_\epsilon = f^{-1}B^n(\epsilon) \, , \quad \delta_\epsilon = \operatorname{cap}\underline{\operatorname{dens}}(E_\epsilon, 0) \, , \tag{15.9}$$

for $\epsilon > 0$. We are going to prove a theorem, which shows that f has an angular limit 0 provided that the numbers δ_ϵ satisfy either (1) $\liminf \delta_\epsilon > 0$ or (2) $\liminf \delta_\epsilon = 0$ with δ_ϵ tending to 0 sufficiently slowly as $\epsilon \to 0$. A result of this character was proved by J. L. Doob [D] in the case of bounded analytic functions.

15.10. Theorem. *Let* $f\colon \mathbf{H}^n \to G'$ *be a qc mapping,* $\epsilon > 0$, $E_\epsilon = f^{-1}B^n(\epsilon)$, *and* $\delta_\epsilon = \operatorname{cap}\underline{\operatorname{dens}}(E_\epsilon, 0)$. *If*

$$\limsup_{\epsilon\to 0} \delta_\epsilon \left(\log \frac{1}{\epsilon}\right)^{n-1} = \infty ,$$

then f *has an angular limit* 0 *at the origin.*

Proof. Suppose, on the contrary, that there exist $\varphi \in (0, \frac{1}{2}\pi)$ and a sequence (b_k) in $C(\varphi)$ with $b_k \to 0$ and $f(b_k) \to \beta \neq 0$ as $k \to \infty$. Let $0 < 2r_0 < |\beta|$. By relabeling if necessary we may assume, in view of 14.5, that $fD(b_k, M) \subset \mathbf{R}^n \setminus \overline{B}^n(r_0)$, $k = 1, 2, \ldots$ for some $M > 0$.

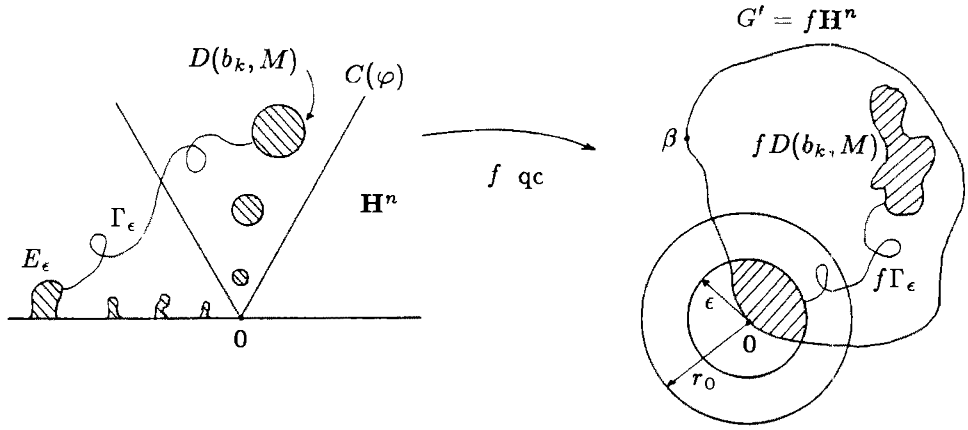


Diagram 15.1. The proof of 15.10.

For every $\epsilon \in (0, r_0)$ there exists t_ϵ such that

$$M(E_\epsilon, r, 0) \geq \tfrac{1}{2}\delta \quad \text{for} \quad r \in (0, t_\epsilon) . \tag{15.11}$$

Fix $\epsilon \in (0, r_0)$. For $|b_k| < t_\epsilon$ denote

$$F_1^k = \overline{B}^n(|b_k|) \cap E_\epsilon , \quad F_2^k = \overline{B}^n(|b_k|) \cap \left(\bigcup D(b_k, M)\right) ,$$
$$F_3^k = S^{n-1}(2|b_k|) \quad , \quad \Gamma_{ij}^k = \Delta(F_i^k, F_j^k; \mathbf{R}^n) .$$

By (15.11) we have for $|b_k| < t_\epsilon$

$$\mathsf{M}(\Gamma_{13}^k) \geq \tfrac{1}{2}\delta_\epsilon .$$

From 14.5 and 5.34 it follows that

$$\mathsf{M}(\Gamma_{23}^k) \geq c(n,\varphi,M) = c > 0$$

for all k. Let $\Gamma_\epsilon = \Delta\big(E_\epsilon, \bigcup D(b_k,M); \mathbf{H}^n\big)$. By virtue of the symmetry principle 5.22 and the comparison principle 5.41 one obtains

$$\mathsf{M}(\Gamma_\epsilon) \geq \tfrac{1}{2}\,\mathsf{M}(\Gamma_{12}^k) \geq \tfrac{1}{2}\cdot 3^{-n}\min\{\tfrac{1}{2}\delta_\epsilon,\ c,\ c_n\log 2\} \geq A\delta_\epsilon \tag{15.12}$$

for $|b_k| < t_\epsilon$ where A is a positive number depending only on n, φ, and M. From (5.14) we get the upper bound

$$\mathsf{M}(f\Gamma_\epsilon) \leq \omega_{n-1}\Big(\log\frac{r_0}{\epsilon}\Big)^{1-n}.$$

This inequality, together with (15.12) and $\mathsf{M}(\Gamma_\epsilon) \leq K_O(f)\,\mathsf{M}(f\Gamma_\epsilon)$, yields

$$A\delta_\epsilon \leq K_O(f)\,\omega_{n-1}\Big(\log\frac{r_0}{\epsilon}\Big)^{1-n}.$$

Letting $\epsilon \to 0$ we get a contradiction. ❐

15.13. Remarks. (1) Theorem 15.1 is a special case of the above result 15.10 when $\liminf \delta_\epsilon > 0$.

(2) Theorems 15.1 and 15.10 seem to be among the best results implying the existence of an angular limit, even in the particular case when f is conformal and $n = 2$ (cf. [VU2], [VU3]).

15.14. An open problem. For $E \subset \mathbf{H}^2$, $0 \in \overline{E}$, and $\alpha \in \mathbf{R}^2$ let $\mathcal{C}(E,\alpha)$ be the class of all conformal mappings of $\mathbf{H}^2$ into $\mathbf{B}^2$ having limit α at 0 through the set E. Assume now that E has the following property:

(15.15) If $f \in \mathcal{C}(E,\alpha)$ then f has angular limit α at 0.

In particular, if $\operatorname{cap}\underline{\operatorname{dens}}(E,0) > 0$, then E has this property by 15.1. Denote $E_M = \bigcup_{x\in E} D(x,M)$, $M > 0$. Does it follow from (15.15) that $\operatorname{cap}\overline{\operatorname{dens}}(E_M,0) > 0$ for some $M > 0$?

15.16. An open problem. For $q \geq 1$ let

$$T_q = \{\, x \in \mathbf{H}^n : x_n \geq (x_1^2 + \ldots + x_{n-1}^2)^{q/2} \,\}$$

and $T_q(z) = T_q + \{z\}$ for $z \in \mathbf{R}^n$ with $z_n = 0$. Let $f\colon \mathbf{H}^n \to \mathbf{R}^n$ be a qc mapping and let $q > 1$ be given. Does there exist $z \in \partial\mathbf{H}^n$ such that f has a limit along $T_q(z)$ at z? In the limiting case $q = 1$ this is true by 14.7. See also 14.4. If $n = 2$, $q = 2$, one can construct a conformal mapping of $\mathbf{H}^2$ having an angular limit α at a single boundary point 0 but failing to have limit α at 0 through T_2 [GAP]. (It is well known that the answer is negative in the case of bounded analytic functions, $n = 2$ (see [CL, p. 43]).)

16. Dirichlet–finite mappings

The goal of this section is to extend Theorem 15.1 so that it applies to coordinate functions f_j, $1 \le j \le n$, of a qc mapping $f\colon \mathbf{H}^n \to G'$, $f = (f_1, \ldots, f_n)$. Such an extension is motivated by a result of F. W. Gehring and A. J. Lohwater [GL], which reads as follows. Let $f\colon \mathbf{H}^2 \to \mathbf{R}^2$, $f = (f_1, f_2)$, be a bounded analytic function, let γ_j be a curve in $\mathbf{H}^2$ terminating at 0, and let f_j have a limit α_j along γ_j, $j = 1, 2$. Then f has an angular limit $\alpha = (\alpha_1, \alpha_2)$ at 0.

It follows from an example due to S. Rickman [RI5] that a similar result is not true for bounded qr mappings in $\mathbf{H}^n$, if $n \ge 3$. In the present section we shall show that the result in [GL] has a counterpart for quasiregular mappings with a finite Dirichlet integral.

Let $u\colon \mathbf{H}^n \to \mathbf{R}$ be a continuous ACL^n function. Then u is said to be *Dirichlet finite*, or to have a *finite Dirichlet integral*, if

$$\text{(16.1)} \qquad \mathrm{Dir}(u) = \int_{\mathbf{H}^n} |\nabla u|^n \, dm < \infty \, .$$

We say that u has a *locally bounded Dirichlet integral* if there exist numbers $B > 0$, $M > 0$ such that

$$\text{(16.2)} \qquad \int_{D(x,M)} |\nabla u|^n \, dm \le B$$

for all $x \in \mathbf{H}^n$ where $D(x, M)$ is as in (2.11).

Let $G \subset \mathbf{R}^n$ be an open set. A continuous function $u\colon G \to \mathbf{R}$ is said to be *monotone* (in the sense of Lebesgue) if the equalities

$$\text{(16.3)} \qquad \max_{\overline{D}} u(x) = \max_{\partial D} u(x) \quad \text{and} \quad \min_{\overline{D}} u(x) = \min_{\partial D} u(x)$$

hold whenever D is a domain with compact closure $\overline{D} \subset G$.

16.4. Remark. It follows from the above definition that if $t \in \mathbf{R}$, then each component $A \neq \emptyset$ of the set $\{ z \in G : u(z) > t \}$ fails to be relatively compact, i.e. $\overline{A} \cap \partial G \neq \emptyset$. A similar statement holds if $>$ is replaced by $\geq$, $<$, or $\leq$. Hence monotone functions satisfy a weak maximum principle. The class of monotone functions is wide: it contains harmonic functions as well as solutions of certain elliptic partial differential equations associated with qr mappings.

16.5. Exercise. (1) The function $u: \mathbf{H}^2 \to (0, \pi)$, $u(z) = \arg z$, is a monotone ACL^2 function. Show by computation that u fails to satisfy (16.1) but that it does satisfy (16.2).

(2) Construct a monotone function $u: \mathbf{H}^2 \to \mathbf{R}_+$ which has no asymptotic value at any point $z \in \partial \mathbf{H}^2$.

The next result is a fundamental property of functions with locally bounded Dirichlet integral. Some results of this kind were proved already by D. Hilbert and H. Lebesgue in the beginning of this century (see [LF1] and the references given there). These ideas have also found frequent application in geometric function theory in connection with the so-called length-area method. For further references see 5.72.

16.6. Theorem. *Let $u: \mathbf{B}^n \to \mathbf{R}$ be a monotone function with locally bounded Dirichlet integral. Then*

$$|u(x) - u(y)|^n \leq C \left(\log \frac{1}{r} \right)^{-1} (1 - r)^{1-n}$$

where $r = \operatorname{th} \frac{1}{4} \rho(x, y)$ and C is a positive constant depending only on the numbers n, M, and B in (16.2). In particular, $u : (\mathbf{B}^n, \rho) \to (\mathbf{R}, | \, |)$ is uniformly continuous.

Proof. Clearly we may assume that $u(x) < u(y)$. Since the right side depends on x and y only through the Möbius invariant quantity $\rho(x, y)$, we may assume that $x = re_1 = -y$, $r = \operatorname{th} \frac{1}{4} \rho(x, y)$ (see (2.25)). Let

$$E = \{ z \in \mathbf{B}^n : u(z) \leq u(x) \} \ , \quad F = \{ z \in \mathbf{B}^n : u(z) \geq u(y) \} \ ,$$

and denote $\Gamma_r = \Delta\big(E, F; B^n(\sqrt{r})\big)$. It follows from 16.4 and 5.32 that

$$\mathsf{M}(\Gamma_r) > c_n \log \frac{1}{\sqrt{r}} \ .$$

Lemma 7.4 yields

$$\mathsf{M}(\Gamma_r) \leq |u(x) - u(y)|^{-n} \int_{B^n(\sqrt{r})} |\nabla u|^n dm \ .$$

In view of (16.2) the integral can be estimated from above in terms of B and the number

$$\inf\{\, k \,:\, \overline{B}^n(\sqrt{r}\,) \subset \bigcup_{j=1}^{k} D(x_j, M)\,,\ |x_j| \le \sqrt{r}\,\}\,.$$

It follows from (4.19) and 4.22 that we now obtain

$$\int_{\overline{B}^n(\sqrt{r}\,)} |\nabla u|^n dm \le Bd(n)(1-\sqrt{r}\,)^{1-n} \le 2^{n-1}Bd(n)(1-r)^{1-n}\,.$$

In conclusion, the above inequalities yield

$$|u(x)-u(y)|^n \le C\left(\log\frac{1}{r}\right)^{-1}(1-r)^{1-n}$$

where $C = 2^n Bd(n)/c_n$. ❐

16.7. Corollary. *Let* $u\colon \mathbf{B}^n \to \mathbf{R}$ *be a monotone* ACL^n *function. Then*

$$|u(x)-u(y)|^n \le \mathrm{Dir}(u)\Big/\left(c_n \log\frac{1}{r}\right)$$

for all $x, y \in \mathbf{B}^n$, *where* c_n *is as in* 5.34 *and* $r = \operatorname{th}\frac{1}{4}\rho(x,y)$.

Proof. Clearly we may assume that $\mathrm{Dir}(u) < \infty$ and $u(x) < u(y)$. Define the sets E and F as in the proof of 16.6 and let $\Gamma = \Delta(E, F; \mathbf{B}^n)$. By 8.6 and 8.7

$$\begin{aligned}\mathsf{M}(\Gamma) &\ge \lambda_{\mathbf{H}^n}(x,y) \ge \tfrac{1}{2}\tau\left(\operatorname{sh}^2 \tfrac{1}{2}\rho(x,y)\right)\\ &\ge -c_n \log \operatorname{th} \tfrac{1}{4}\rho(x,y)\,.\end{aligned}$$

Lemma 7.6 yields

$$\mathsf{M}(\Gamma) \le |u(x)-u(y)|^{-n}\,\mathrm{Dir}(u)$$

and hence the result follows. ❐

We remark that the upper bound in 16.6 or 16.7 is not accurate for large values of $\rho(x,y)$. A better estimate for large values of $\rho(x,y)$ can be derived from 16.6 and the fact that u is uniformly continuous, see 4.13.

16.8. Theorem. *Let* $u\colon \mathbf{H}^n \to \mathbf{R}$ *be a monotone Dirichlet finite function and let* $E \subset \mathbf{H}^n$ *be a set with* $0 \in \overline{E} \subset \mathbf{H}^n \cup \{0\}$ *and* $\operatorname{cap}\underline{\operatorname{dens}}(E,0) > 0$. *If* $u(x) \to \alpha$ *as* $x \to 0$, $x \in E$, *then* u *has an angular limit* α *at* 0.

Proof. The proof is similar to that of 15.1. Fix $\varphi \in (0, \frac{1}{2}\pi)$. Suppose, on the contrary, that there exists a sequence (b_k) in $C(\varphi)$ with $b_k \to 0$ and $u(b_k) \to \beta \neq \alpha$. We shall assume that $-\infty < \alpha < \beta < \infty$; in other cases the proof is similar. Let B_k be the b_k-component of the set $B = \{ z \in \mathbf{H}^n : u(z) > (\alpha + 2\beta)/3 \}$ and let $A = \{ z \in \mathbf{H}^n : u(z) < (2\alpha + \beta)/3 \}$. By 16.7 and the proof of 14.5 there are $M > 0$ and $p \in \mathbf{N}$ such that $D(b_k, M) \subset B_k$ for all $k \geq p$ and

$$\operatorname{rad} \overline{\operatorname{dens}}(B, 0) \geq d(\varphi, M) > 0 ;$$

hence $\operatorname{cap} \overline{\operatorname{dens}}(B, 0) > 0$ by 14.11. Since

$$\operatorname{cap} \underline{\operatorname{dens}}(A, 0) \geq \operatorname{cap} \underline{\operatorname{dens}}(E, 0) > 0$$

it follows from 14.14 and 5.22 that

$$\mathsf{M}(\Delta(A, B; \mathbf{H}^n)) \geq \tfrac{1}{2} \mathsf{M}(\Delta(A, B; \mathbf{R}^n)) = \infty .$$

From 7.6 we have

$$\mathsf{M}(\Delta(A, B; \mathbf{H}^n)) \leq 3^n (\beta - \alpha)^{-n} \operatorname{Dir}(u) < \infty ,$$

which is a contradiction. ❐

16.9. Corollary. *Let $f \colon \mathbf{H}^n \to \mathbf{R}^n$ be a qr mapping and assume that there are sets $E_j \subset \mathbf{H}^n$ such that $f_j(x) \to \alpha_j$ as $x \to 0$, $x \in E_j$, $j = 1, \dots, n$. If $\operatorname{cap} \underline{\operatorname{dens}}(E_j, 0) > 0$ and $\operatorname{Dir}(f_j) < \infty$ for each $j = 1, \dots, n$, then f has an angular limit $\alpha = (\alpha_1, \dots, \alpha_n)$ at 0.*

Proof. The proof follows from 16.5(2) and 16.8. ❐

16.10. Corollary. *Let $f \colon \mathbf{H}^n \to \mathbf{R}^n$ be a qc mapping and assume that $f_j(x) \to \alpha_j$ as $x \to 0$, $x \in E_j$, $E_j \subset \mathbf{H}^n$, $j = 1, \dots, n$. If $\operatorname{cap} \underline{\operatorname{dens}}(E_j, 0) > 0$, $j = 1, \dots, n$, then f has an angular limit $\alpha = (\alpha_1, \dots, \alpha_n)$ at 0.*

Proof. Let $h \in \mathcal{M}(\mathbf{R}^n)$ be such that $h(e_n) = \infty$ and $hD(e_n, 1) = \overline{\mathbf{R}}^n \setminus \mathbf{B}^n$. By considering the map $h \circ f$, if necessary, we may assume that f is bounded by 1 in $\mathbf{H}^n \cap B^n(\frac{1}{4}) = D$ (note: here we use the fact that f is injective). Moreover,

$$\int_D |f'(x)|^n dm \leq K \int_D J_f(x)\, dm = K\, m(fD) < K\, \Omega_n .$$

Since $|\partial f_j(x)/\partial x_k| \leq |f'(x)|$, $1 \leq i, k \leq n$, we see that $\operatorname{Dir}(f_j) < n^n K\, \Omega_n$, $j = 1, \dots, n$, and hence the proof follows from 16.8. ❐

It follows from a well-known formula for change of variables that all K-qr mappings $f\colon \mathbf{H}^n \to \mathbf{B}^n$ have a finite Dirichlet integral, that is $\mathrm{Dir}(f) \le Km(\mathbf{B}^n) = K\Omega_n$. More generally this holds for K-qr mappings $f\colon \mathbf{H}^n \to \mathbf{B}^n$ with finite maximal multiplicity $N(f,\mathbf{H}^n) < \infty$, that is $\mathrm{Dir}(f) \le KN(f,\mathbf{H}^n)\,\Omega_n$. It is easy to give examples of bounded analytic functions with an infinite Dirichlet integral (for instance, the exponential function in the left plane). However, bounded qr mappings have a locally bounded Dirichlet integral according to the following theorem of Reshetnyak (proof omitted) [R13, p. 127].

16.11. Theorem. *For* $n \ge 2$, $K \ge 1$, and $r \in (0,1)$ *there exists a number* $c(n,K,r)$ *such that each* K*-qr mapping* $f\colon \mathbf{B}^n \to \mathbf{B}^n$ *satisfies*

$$\int_{B^n(r)} |f'(x)|^n dm \le c(n,K,r)\ .$$

16.12. Theorem. *Let* $f\colon \mathbf{B}^n \to \mathbf{R}^n$ *be qr. Then the following conditions are equivalent:*

(1) $f\colon (\mathbf{B}^n,\rho) \to (\mathbf{R}^n, |\ |)$ *is uniformly continuous.*

(2) *There are numbers* $M > 0$ *and* $B > 0$ *such that*

$$\int_{D(z,M)} |f'(x)|^n dm \le B$$

holds for every $z \in \mathbf{B}^n$.

(3) *There are numbers* $T > 0$ *and* C *such that* $|f(x) - f(y)| \le C$ *whenever* $x, y \in \mathbf{B}^n$ *and* $\rho(x,y) \le T$.

Proof. (1) $\Rightarrow$ (2): Fix $t > 0$ such that $\rho(x,y) \le t$ implies $|f(x) - f(y)| \le 1$. It follows from (2.23) that

$$B^n\big(x,\ (1-|x|)\,\mathrm{th}\,\tfrac{1}{2}t\big) \subset D(x,t)\ .$$

Let $h \in \mathcal{M}(\mathbf{R}^n)$ with $h\mathbf{B}^n = B^n\big(x,\ (1-|x|)\,\mathrm{th}\,\tfrac{1}{2}t\big)$. Then $f \circ h\colon \mathbf{B}^n \to \mathbf{B}^n$ is K-qr and

$$\int_{B^n(1/2)} |(f\circ h)'|^n dm = \int_{B^n(x,s)} |f'|^n dm \le c(n,K,\tfrac{1}{2})$$

by 16.11, where $s = \frac{1}{2}(1-|x|)\,\mathrm{th}\,\frac{1}{2}t$. Now choose a number M such that $D(x,M) \subset B^n(x,s)$ for all x. Because for all $x \in \mathbf{B}^n$

$$D(x,M) \subset B^n\big(x,\ (e^M - 1)(1-|x|)\,\big)$$

by (2.23), the choice $e^M - 1 = \frac{1}{2}\operatorname{th}\frac{1}{2}t$ implies that $D(x,M) \subset B^n(x,s)$. With this choice of M and with $B = c(n,K,\frac{1}{2})$ the condition (2) holds.

(2) $\Rightarrow$ (1) : It suffices to show that each coordinate function of f is uniformly continuous. On the other hand the coordinate functions are monotone. Now the uniform continuity of coordinate functions follows from (2) and 16.6.

It is clear that (1) $\Rightarrow$ (3). So it remains to prove (3) $\Rightarrow$ (1). It follows from the Schwarz lemma 11.2 (see also the proof of 13.4) that

$$|f(x) - f(y)| \le a(n,K)\left(\frac{\operatorname{th}\frac{1}{2}\rho(x,y)}{\operatorname{th}\frac{1}{2}T}\right)^{\alpha},$$

where $\alpha = K_I(f)^{1/(1-n)}$. The desired conclusion follows. □

16.13. Corollary. *Suppose that* $f\colon \mathbf{B}^n \to \mathbf{R}^n$ *is a quasiregular mapping with* $\int_{\mathbf{B}^n} |f'(x)|^n dm < \infty$. *Then*

$$|f(x)| \le |f(0)| + 1 + \frac{1}{M}\log\frac{1+|x|}{1-|x|}$$

for all $x \in \mathbf{B}^n$ *where* $M = \sup\{T : \rho(x,y) \le T \Rightarrow |f(x) - f(y)| \le 1\}$.

Proof. The proof follows from the proof of 4.13 and from 16.12. □

16.14. Remark. Theorem 16.11 yields an upper bound for the growth of the Dirichlet integral of a bounded K–qr mapping $f\colon \mathbf{B}^n \to \mathbf{B}^n$. Indeed, 16.11 combined with 4.22 shows that

$$\int_{B^n(r)} |f'(x)|^n dm \le A(1-r)^{1-n},$$

where A depends only on n and K.

16.15. Notes. The results of this section are from [VU9] except for 16.12 which is [VU10, 4.29].

Some open problems

(1) Find an explicit expression for $\gamma_n(s)$ when $n \geq 3$ (see Sections 5 and 7).

(2) Let E, $F \subset \mathbf{H}^n$ be compact and disjoint, let $F^* = \{(x_1, \ldots, x_{n-1}, -x_n) : (x_1, \ldots, x_n) \in F\}$, $\Gamma = \Delta(E, F)$, $\Gamma^* = \Delta(E, F^*)$. Is it true that $\mathsf{M}(\Gamma) \geq \mathsf{M}(\Gamma^*)$ (cf. 7.59)?

(3) Find all domains D such that $\lambda_D(x, y)^{1/(1-n)}$ is a metric on D. Is this true for $D = \mathbf{R}^n \setminus \{0\}$ and $n = 2$ (cf. Section 8)?

(4) Let $f\colon \mathbf{B}^n \to f\mathbf{B}^n \subset \mathbf{B}^n$ be discrete, open, and proper. Assume that $n \geq 3$ and B_f is compact. Is f one-to-one (Section 9)? The answer is yes if $f\mathbf{B}^n = \mathbf{B}^n$.

(5) Find an upper bound for the linear dilatation $H(x, f)$ of a K-qc mapping $f\colon G \to fG$, $G \subset \mathbf{R}^n$, such that the bound tends to 1 as $K \to 1$ (cf. Section 10).

(6) Does there exist an absolute constant C, independent of n and K, such that Theorem 11.40 holds with C in place of $M_1(n, K)$?

(7) For given $n \geq 2$, $K \geq 1$, and $\delta \in (0, 1)$, does there exist a number $A(n, K, \delta)$ with the following property: if $f\colon \mathbf{B}^n \to f\mathbf{B}^n \subset \mathbf{B}^n$ is K-qr and $|f(0)| \geq \delta$ then

$$\operatorname{card}\{\, z \in B^n(\tfrac{1}{2}) : f(z) = 0 \,\} \leq A \,?$$

(8) Let $f\colon \mathbf{B}^n \to \mathbf{B}^n$, $n \geq 3$, be qr. Show that f has at least one radial limit. (The case of Dirichlet-finite f is well known [MIK2], [MR1].)

(9) Prove or disprove the following assertion. For each $n \geq 2$, $r \in (0, 1)$, and $K \geq 1$ there exists a number $d(n, K, r)$ with $d(n, K, r) \to d(n, K)$ as $r \to 0$ and $d(n, K) \to 1$ as $K \to 1$ such that whenever $f\colon \mathbf{B}^n \to \mathbf{R}^n$ is K-qc, then $fB^n(r)$ is a $d(n, K, r)$-quasiball. More precisely, the representation $fB^n(r) = g\mathbf{B}^n$ holds where $g\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ is a $d(n, K, r)$-qc mapping with $g(\infty) = \infty$. (Note: It was kindly pointed out by J. Becker that we can choose $d(2, 1, r) = (1 + r)/(1 - r)$ either by [BC, pp. 39–40] or by a more general result of S. L. Krushkal' [KR].)

Additional open problems can be found in [BAM], [G9], and [V10].

Bibliography

[AS] M. ABRAMOWITZ and I. STEGUN: *Handbook of mathematical functions with formulas, graphs and mathematical tables.* – National Bureau of Standards, 1964 (Russian translation, Nauka 1979).

[AG] S. AGARD: *Distortion theorems for quasiconformal mappings.* – Ann. Acad. Sci. Fenn. Ser. A I 413 (1968), 1–12.

[AM] S. AGARD and A. MARDEN: *A removable singularity theorem for local homeomorphisms.* – Indiana Math. J. 20 (1970), 455–461.

[A1] L.V. AHLFORS: *On quasiconformal mappings.* – J. Analyse Math. 3 (1954), 1–58, 207–208.

[A2] L.V. AHLFORS: *Lectures on quasiconformal mappings.* – Van Nostrand, 1966.

[A3] L.V. AHLFORS: *Conformal invariants.* – McGraw–Hill, New York, 1973.

[A4] L.V. AHLFORS: *Quasiconformal mappings, Teichmüller spaces and Kleinian groups.* – Proc. Internat. Congr. Math. (Helsinki, 1978), Vol. 1, 71–84, Acad. Sci. Fennica, Helsinki, 1980.

[A5] L.V. AHLFORS: *Möbius transformations in several dimensions.* – School of Mathematics, University of Minnesota, 1981.

[AB] L.V. AHLFORS and A. BEURLING: *Conformal invariants and function–theoretic null–sets.* – Acta Math. 83 (1950), 101–129.

[AN1] G.D. ANDERSON: *Extremal rings in n–space for fixed and varying n.* – Ann. Acad. Sci. Fenn. Ser. A I 575 (1974), 1–21.

[AN2] G.D. ANDERSON: *Dependence on dimension of a constant related to the Grötzsch ring.* – Proc. Amer. Math. Soc. 61 (1976), 77–80.

[AV] G.D. ANDERSON and M.K. VAMANAMURTHY: *Hölder continuity of quasiconformal mappings of the unit ball.* – Proc. Amer. Math. Soc. (to appear).

[AVV1] G.D. ANDERSON, M.K. VAMANAMURTHY, and M. VUORINEN: *Dimension–free quasiconformal distortion in n–space.* – Trans. Amer. Math. Soc. 297 (1986), 687–706.

[AVV2] G.D. ANDERSON, M.K. VAMANAMURTHY, and M. VUORINEN: *Sharp distortion theorems for quasiconformal mappings.* – Trans. Amer. Math. Soc. 305 (1988), 95–111.

[AVV3] G.D. ANDERSON, M.K. VAMANAMURTHY, and M. VUORINEN: *Special functions of quasiconformal theory.* – In preparation.

[AVV4] G.D. ANDERSON, M.K. VAMANAMURTHY, and M. VUORINEN: *Inequalities for the extremal distortion function.* – Proc. 13th Rolf Nevanlinna Colloquium, Joensuu Aug. 1987, Springer-Verlag (to appear).

[ASY] V.V. ASEEV and A.V. SYCHEV: *On the removable sets of spatial quasiconformal mappings.* (Russian). – Sibirsk. Mat. Zh. 15 (1974), 1213–1227.

[AST] K. ASTALA: *H^p–theory for quasiconformal mappings.* – (to appear).

[ASTG] K. ASTALA and F.W. GEHRING: *Injectivity, the BMO norm and the universal Teichmüller space.* – J. Analyse Math. 46 (1986), 16–57.

[BAM] A. BAERNSTEIN and J. J. MANFREDI: *Topics in quasiconformal mapping.* – Topics in modern harmonic analysis. Proc. Seminar held in Torino and Milano, May–June 1982, Vol. II, 849–862.

[B] T. BAGBY: *The modulus of a plane condenser.* – J. Math. Mech. 17 (1967), 315–329.

[BS] F. BAGEMIHL and W. SEIDEL: *Sequential and continuous limits of meromorphic functions.* – Ann. Acad. Sci. Fenn. Ser. A I 280 (1960), 1–17.

[BA] C. BANDLE: *Isoperimetric inequalities and applications.* – Pitman, 1980.

[BBH] K. F. BARTH, D. A. BRANNAN, and W. K. HAYMAN: *Research problems in complex analysis.* – Bull. London Math. Soc. 16 (1984), 490–517.

[BE] A. F. BEARDON: *The geometry of discrete groups.* – Graduate Texts in Math. Vol. 91, Springer-Verlag, Berlin–Heidelberg–New York, 1982.

[BC] J. BECKER: *Löwnersche Differentialgleichung und quasiconform fortsetzbare schlichte Funktionen.* – J. Reine Angew. Math. 255 (1972), 23–43.

[BEL] P. P. BELINSKIĬ: *General properties of quasiconformal mappings.* (Russian). – Izdat. "Nauka", Sibirsk. Otdelenie, Novosibirsk, 1974.

[BER] M. BERGER: *Geometry* I, II. – Springer-Verlag, Berlin–Heidelberg–New York, 1987.

[BI1] B. BOJARSKI and T. IWANIEC: *Another approach to Liouville theorem.* – Math. Nachr. 107 (1982), 253–262.

[BI2] B. BOJARSKI and T. IWANIEC: *Analytical foundations of the theory of quasiconformal mappings in* R^n. – Ann. Acad. Sci. Fenn. Ser. A I Math. 8 (1983), 257–324.

[BB] J.M. BORWEIN and P.B. BORWEIN: *Pi and the AGM*. – John Wiley & Sons, Inc., New York, 1987.

[BU] R.B. BURCKEL: *An introduction to classical complex analysis.* Vol. I. – Academic Press, 1979.

[BF] P.F. BYRD and M.D. FRIEDMAN: *Handbook of elliptic integrals for engineers and physicists.* – Die Grundlehren der math. Wissenschaften Vol. 57, Springer-Verlag, Berlin–Göttingen–Heidelberg, 1954.

[CAL] E.D. CALLENDER: *Hölder continuity of* n*-dimensional quasiconformal mappings.* – Pacific J. Math. 19 (1960), 499–515.

[C1] P. CARAMAN: n*-dimensional quasiconformal* (QCf) *mappings.* – Editura Academiei Române, Bucharest, Abacus Press, Tunbridge Wells Haessner Publishing, Inc., Newfoundland, New Jersey, 1974.

[C2] P. CARAMAN: *About a conjecture of F. W. Gehring on the boundary correspondence by quasiconformal mappings.* – Ann. Polon. Math. 33 (1976/77), 21–33.

[CA] C. CARATHÉODORY: *Theory of functions of a complex variable I, II.* – Chelsea Publ. Co., New York, 1954.

[CHE1] A.V. CHERNAVSKIĬ: *Finite-to-one open mappings of manifolds.* (Russian). – Mat. Sb. 65 (1964), 357–369.

[CHE2] A. V. CHERNAVSKIĬ: *Remarks on the paper "On finite-to-one mappings of manifolds".* (Russian). – Mat. Sb. 66 (1965), 471–472.

[CH] P.T. CHURCH and E. HEMMINGSEN: *Light open maps on* n*-manifolds.* – Duke Math. J. 27 (1960), 527–536.

[CL] E.F. COLLINGWOOD and A.J. LOHWATER: *The theory of cluster sets.* – Cambridge Tracts in Mathematics and Mathematical Physics, No. 56, Cambridge University Press, 1966.

[CG] H.S.M. COXETER and S.L. GREITZER: *Geometry revisited.* – The Math. Association of America, 1976.

[DE] K. DEIMLING: *Nichtlineare Gleichungen und Abbildungsgrade.* – Springer-Verlag, Berlin–Heidelberg–New York, 1974.

[D] J.L. DOOB: *The boundary values of analytic functions.* – Trans. Amer. Math. Soc. 34 (1932), 153–170.

[FA] K.J. FALCONER: *The geometry of fractal sets.* – Cambridge University Press, 1985.

[FV] R. FEHLMANN and M. VUORINEN: *Mori's theorem for n–dimensional quasiconformal mappings.* – Ann. Acad. Sci. Fenn. Ser. A I Math. 13 (1988) (to appear).

[FE] J. FERRAND: *A characterization of quasiconformal mappings by the behaviour of a function of three points.* – Proc. 13th Rolf Nevanlinna Colloquium, Joensuu Aug. 1987, Springer-Verlag (to appear).

[FO] L.R. FORD: *Automorphic functions.* – Second ed., Chelsea, New York, 1972.

[FR] J. FREHSE: *Capacity methods in the theory of partial differential equations.* – Jber. d. Dt. Math.–Verein 84 (1982), 1–44.

[F] B. FUGLEDE: *Extremal length and functional completion.* – Acta Math. 98 (1957), 171–219.

[GAP] D. GAIER and CH. POMMERENKE: *On the boundary behavior of conformal maps.* – Michigan Math. J. 14 (1967), 79–82.

[GÁL] I.S. GÁL: *Conformally invariant metrics and uniform structures.* – Indag. Math. 22 (1960), 218–244.

[G1] F.W. GEHRING: *Symmetrization of rings in space.* – Trans. Amer. Math. Soc. 101 (1961), 499–519.

[G2] F.W. GEHRING: *Rings and quasiconformal mappings in space.* – Trans. Amer. Math. Soc. 103 (1962), 353–393.

[G3] F.W. GEHRING: *The Carathéodory convergence theorem for quasiconformal mappings in space.* – Ann. Acad. Sci. Fenn. Ser. A I 336/11 (1963), 1–21.

[G4] F.W. GEHRING: *Quasiconformal mappings in space.* – Bull. Amer. Math. Soc. 69 (1963), 146–164.

[G5] F.W. GEHRING: *The L^p–integrability of the partial derivatives of quasiconformal mappings.* – Acta Math. 130 (1973), 265–277.

[G6] F.W. GEHRING: *A remark on domains quasiconformally equivalent to a ball.* – Ann. Acad. Sci. Fenn. Ser. A I Math. 2 (1976), 147–155.

[G7] F.W. GEHRING: *Quasiconformal mappings.* – Complex Analysis and its Applications, Vol. II, International Atomic Energy Agency, Vienna, 1976.

[G8] F.W. GEHRING: *Characteristic properties of quasidisks.* – Les Presses de l'Universite de Montreal, Montreal, 1982.

[G9] F.W. GEHRING: *Topics in quasiconformal mappings.* – Proc. Internat. Congr. Math. (Berkeley, California, 1986).

[G10] F.W. GEHRING: *Uniform domains and the ubiquitous quasidisk.* – Jber. d. Dt. Math.–Verein 89 (1987), 88–103.

[GL] F.W. GEHRING and A.J. LOHWATER: *On the Lindelöf theorem.* – Math. Nachr. 19 (1958), 165–170.

[GMA] F. W. GEHRING and G. J. MARTIN: *Discrete quasiconformal groups I.* – Proc. London Math. Soc. (3) 55 (1987), 331–358.

[GM1] F.W. GEHRING and O. MARTIO: *Quasiextremal distance domains and extension of quasiconformal mappings.* – J. Analyse Math. 45 (1985), 181–206.

[GM2] F.W. GEHRING and O. MARTIO: *Lipschitz classes and quasiconformal mappings.* – Ann. Acad. Sci. Fenn. Ser. A I Math. 10 (1985), 203–219.

[GOS] F.W. GEHRING and B.G. OSGOOD: *Uniform domains and the quasi-hyperbolic metric.* – J. Analyse Math. 36 (1979), 50–74.

[GP] F.W. GEHRING and B.P. PALKA: *Quasiconformally homogeneous domains.* – J. Analyse Math. 30 (1976), 172–199.

[GV1] F.W. GEHRING and J. VÄISÄLÄ: *The coefficients of quasiconformality of domains in space.* – Acta Math. 114 (1965), 1–70.

[GV2] F.W. GEHRING and J. VÄISÄLÄ: *Hausdorff dimension and quasiconformal mappings.* – J. London Math. Soc. 6(2) (1973), 504–512.

[GIA] M. GIAQUINTA: *Multiple integrals in the calculus of variations and nonlinear elliptic systems.* – Ann. of Math. Studies No. 105, Princeton Univ. Press, Princeton, New Jersey, 1983.

[GT] D. GILBARG and N.S. TRUDINGER: *Elliptic partial differential equations of second order.* – Die Grundlehren der math. Wissenschaften Vol. 224, Berlin–Heidelberg–New York, Springer-Verlag, 1977.

[GOR] V.M. GOL'DSTEIN and YU. G. RESHETNYAK: *Introduction to the theory of functions with generalized derivatives and quasiconformal mappings.* (Russian). – Izdat. "Nauka", Moscow, 1983.

[GR] S. GRANLUND: *Harnack's inequality in the borderline case.* – Ann. Acad. Sci. Fenn. Ser. A I Math. 5 (1980), 159–163.

[GLM1] S. GRANLUND, P. LINDQVIST, and O. MARTIO: *Conformally invariant variational integrals.* – Trans. Amer. Math. Soc. 277 (1983), 43–73.

[GLM2] S. GRANLUND, P. LINDQVIST, and O. MARTIO: *F–harmonic measure in space.* – Ann. Acad. Sci. Fenn. Ser. A I Math. 7 (1982), 233–247.

[GLM3] S. GRANLUND, P. LINDQVIST, and O. MARTIO: *Phragmén–Lindelöf's and Lindelöf's theorem.* – Ark. Mat. 23 (1985), 103–128.

[GROM] M. GROMOV: *Structures métriques pour les variétés riemanniennes. Rédigé par J. Lafontaine et P. Pansu.* – Cedic, Paris, 1981.

[GRÖ] H. GRÖTZSCH: *Über die Verzerrung bei schlichten nichtkonformen Abbildungen und über eine damit zusammenhängende Erweiterung des Picardschen Satzes.* – Ber. Verh. Sächs. Akad. Wiss. Leipzig 80 (1928), 503–507.

[GUR] L.G. GUROV and YU.G. RESHETNYAK: *A certain analogue of the concept of a function with bounded mean oscillation* (Russian). – Sibirsk. Mat. Zh. 17 (1976), 540–546.

[GU] M. DE GUZMAN: *Differentiation of integrals in* R^n. – Lecture Notes in Math. Vol. 481, Springer-Verlag, Berlin-Heidelberg–New York, 1975.

[H] T. HALL: *Sur la mesure harmonique de certains ensembles.* – Ark. Mat. Astr. Fys. 25A, No 28 (1937), 1–8.

[HA1] W.K. HAYMAN: *Uniformly normal families.* – Lectures on Functions of a Complex Variable, Univ. of Michigan Press, Ann Arbor, 1955, 199–212.

[HA2] W.K. HAYMAN: *Multivalent functions.* – Cambridge Tracts in Mathematics and Mathematical Physics, No. 48, Cambridge University Press, 1958.

[HEL] C.-Q. HE and Z. LI: *Quasiconformal mappings.* – Contemporary Mathematics 48 (1985), 129–150.

[HMA] J. HEINONEN and O. MARTIO: *Estimates for F-harmonic measures and Øksendal's theorem for quasiconformal mappings.* – Indiana Univ. Math. J. 36 (1987), 659–683.

[HM] M. HEINS: *The minimum modulus of a bounded analytic function.* – Duke Math. J. 14 (1947), 170–215.

[HEI] E. HEINZ: *An elementary analytic theory of the degree of mapping in n-dimensional space.* – J. Math. Mech. 8 (1959), 231–247.

[HVU] D.A. HERRON and M. VUORINEN: *Positive harmonic functions in uniform and admissible domains.* – Analysis (to appear).

[HE1] J. HERSCH: *Longueurs extrémales et théorie des fonctions.* – Comment. Math. Helv. 29 (1955), 301–337.

[HE2] J. HERSCH: *Comments on G. Polya's papers 177 and 187.* – G. Polya's Collected Papers, Vol. III, 1985.

[HEP] J. HERSCH and A. PFLUGER: *Généralization du lemme de Schwarz et du principe de la mesure harmonique pour les fonctions pseudo-analytiques.* – C. R. Acad. Sci. Paris 234 (1952), 43–45.

[HES] J. HESSE: *A p-extremal length and p-capacity equality.* – Ark. Mat. 13 (1975), 131–144.

[HS] E. HEWITT and K. STROMBERG: *Real and abstract analysis.* – Springer-Verlag, Berlin-Heidelberg-New York, 1969.

[HCV] D. HILBERT and S. COHN-VOSSEN: *Geometry and the imagination.* – New York, 1952.

[HÜ] O. HÜBNER: *Remarks on a paper by Lawrynowicz on quasiconformal mappings.* – Bull. de L'Acad. Polon. des Sci. 18 (1970), 183–186.

[HW] W. HUREWICZ and H. WALLMAN: *Dimension theory.* – Princeton Univ. Press, Princeton, New Jersey, 1948.

[IK] K. IKOMA: *A modification of Teichmüller's module theorem and its application to a distortion problem in n-space.* – Tôhoku Math. J. 32 (1980), 393–398.

[I] T. IWANIEC: *Some aspects of partial differential equations and quasiregular mappings.* – Proc. Internat. Congr. Math. (Warsaw, 1983), Vol. 2, 1193–1208, PWN, Warsaw, 1984.

[JE1] J.A. JENKINS: *Univalent functions and conformal mapping.* – Ergebnisse der Math. Vol. 18, Corrected ed., Springer-Verlag, Berlin-Heidelberg-New York, 1965.

[JE2] J. A. JENKINS: *The method of extremal metric.* – Proc. Congr. on the occasion of the solution of the Bieberbach conjecture, Amer. Math. Soc. 1986, 95–104.

[J1] P.W. JONES: *Extension theorems for BMO.* – Indiana Univ. Math. J. 29 (1980), 41–66.

[J2] P.W. JONES: *Quasiconformal mappings and extendability of functions in Sobolev spaces.* – Acta Math. 147 (1981), 71–88.

[KL] H. KLOKE: *On the capacity of a plane condenser and conformal mapping.* – J. Reine Angew. Math. 358 (1985), 179–201.

[KO] A.P. KOPYLOV: *On the stability of isometric mappings.* (Russian). – Sibirsk. Math. Zh. 25 (1984), 132–144.

[KR] S.L. KRUSHKAL': *On an interpolating family of a univalent analytic function.* (Russian). – Preprint No 18 (1986), Institute of Mathematics of the Sibirian Branch of the USSR Academy of Sciences, Novosibirsk.

[KK] S.L. KRUSHKAL' and R. KÜHNAU: *Quasiconformal mappings — new methods and applications.* (Russian). – Izdat. "Nauka", Sibirsk. Otdelenie, Novosibirsk, 1984.

[KÜ] H.P. KÜNZI: *Quasikonforme Abbildungen.* – Ergebnisse der Mathematik Vol. 26, Springer-Verlag, Berlin–Göttingen–Heidelberg, 1960.

[K1] T. KUUSALO: *Generalized conformal capacity and quasiconformal metrics.* – Proc. of the Romanian–Finnish Seminar on Teichmüller Spaces and Quasiconformal Mappings, Brasov 1971, pp. 193–202.

[K2] T. KUUSALO: *Hurwitz' theorem for open mappings.* – Preprint 29, 1984, University of Jyväskylä, Finland.

[KU] G.V. KU'ZMINA: *Moduli of families of curves and quadratic differentials.* – Proc. Steklov Institute of Math. 1982, Issue 1 (Russian original: Tom 139, 1980).

[LA] N.S. LANDKOF: *Foundations of modern potential theory.* – Die Grundlehren der math. Wissenschaften Vol. 180, Springer-Verlag, Berlin–New York–Heidelberg, 1972.

[LP] P. LAPPAN: *A non–normal locally uniformly univalent function.* – Bull. London Math. Soc. 5 (1973), 291–242.

[LAV] M.A. LAVRENT'EV: *Sur un critère différentiel des transformations homéomorphes des domaines a trois dimensions.* – Dokl. Akad. Nauk SSSR 22 (1938), 241–242.

[LEVU] M. LEHTINEN and M. VUORINEN: *On Teichmüller's modulus problem in the plane.* – Rev. Roumaine Math. Pures Appl., S. Stoilow anniversary volume (to appear).

[L1] O. LEHTO: *Quasiconformal homeomorphisms and Beltrami equations.* – Discrete Groups and Automorphic Functions (Cambridge, 1975), 121–142, Academic Press, London, 1977.

[L2] O. LEHTO: *A historical survey of quasiconformal mappings.* – On the work of Leonhard Euler (Berlin, 1983), 205–217, Birkhäuser, Basel–Boston,Mass., 1984.

[L3] O. LEHTO: *Univalent functions and Teichmüller spaces.* – Graduate Texts in Math. Vol. 109, Springer-Verlag, Berlin–Heidelberg–New York, 1987.

[LV1] O. LEHTO and K.I. VIRTANEN: *Boundary behaviour and normal meromorphic functions.* – Acta Math. 97 (1957), 47–65.

[LV2] O. LEHTO and K.I. VIRTANEN: *Quasiconformal mappings in the plane.* – Die Grundlehren der math. Wissenschaften Vol. 126, Second ed., Springer-Verlag, Berlin–Heidelberg–New York, 1973.

[LVV] O. LEHTO, K.I. VIRTANEN, and J. VÄISÄLÄ: *Contributions to the distortion theory of quasiconformal mappings.* – Ann. Acad. Sci. Fenn. Ser. A I 273 (1959), 1–14.

[LF1] J. LELONG–FERRAND: *Représentation conforme et transformations a intégrale de Dirichlet bornée.* – Gauthier–Villars, Paris, 1955.

[LF2] J. LELONG-FERRAND: *Invariants conformes globaux sur les varietes riemanniennes.* – J. Differential Geom. 8 (1973), 487–510.

[LF3] J. LELONG-FERRAND: *Construction de modules de continuité dans le cas limite de Soboleff et applications a la géometrie différentielle.* - Arch. Rat. Mech. Anal. 52 (1973), 297–311.

[LF4] J. LELONG-FERRAND: *Problèmes de géométrie conforme.* – Proc. Internat. Congr. Math. (Vancouver, B. C., 1974), Vol. 2, 13–19, Canad. Math. Congress, Montreal, Que., 1975.

[LF5] J. LELONG-FERRAND: *Construction de métriques pour lesquelles les transformations quasiconformes sont lipschitziennes.* – Symposia Mathematica Vol. XVIII (1976), INDAM, Academic Press, London, 407–420.

[LI1] P. LINDQVIST: *On the definition and properties of p-superharmonic functions.* – J. Reine Angew. Math. 365 (1986), 67–79.

[LI2] P. LINDQVIST: *On the quasiregularity of a limit mapping.* – Ann. Acad. Sci. Fenn. Ser. A I Math. 11 (1986), 155-159.

[LIM] P. LINDQVIST and O. MARTIO: *Two theorems of N. Wiener for solutions of quasilinear elliptic equations.* – Acta Math. 155 (1985), 153–171.

[LO] C. LOEWNER: *On the conformal capacity in space.* – J. Math. Mech. 8 (1959), 411–414.

[LOH] A. J. LOHWATER: *The boundary behavior of analytic functions.* (Russian). – In: Mathematical analysis Vol. 10, Itogi Nauki i Tehniki pp. 99–259, Akad. Nauk. SSSR, Vsesojuz Inst. Nauch. i Tehn. Informacii, Moscow, 1973.

[MA] G. MARTIN: *Quasiconformal and bilipschitz mappings, uniform domains and the hyperbolic metric.* – Trans. Amer. Math. Soc. 292 (1985), 169–192.

[MAO] G. MARTIN and B.G. OSGOOD: *The quasihyperbolic metric and the associated estimates on the hyperbolic metric.* – J. Analyse Math. 47 (1986), 37–53.

[M1] O. MARTIO: *A capacity inequality for quasiregular mappings.* – Ann. Acad. Sci. Fenn. Ser. A I 474 (1970), 1–18.

[M2] O. MARTIO: *On the integrability of the derivative of a quasiregular mapping.* – Math. Scand. 35 (1974), 43–48.

[M3] O. MARTIO: *Equicontinuity theorem with an application to variational integrals.* – Duke Math. J. 42 (1975), 569-581.

[M4] O. MARTIO: *Capacity and measure densities.* – Ann. Acad. Sci. Fenn. Ser. A I Math. 4 (1978/1979), 109–118.

[M5] O. MARTIO: *Reflection principle for elliptic partial differential equations and quasiregular mappings.* – Ann. Acad. Sci. Fenn. Ser. A I Math. 6 (1981), 179–188.

[M6] O. MARTIO: *Non-linear potential theory.* – Summer School in Potential theory, Joensuun Korkeakoulu, 1983.

[M7] O. MARTIO: *Counterexamples for unique continuation.* – Manuscripta Math. 60 (1988), 21–47.

[MR1] O. MARTIO and S. RICKMAN: *Boundary behavior of quasiregular mappings.* – Ann. Acad. Sci. Fenn. Ser. A I 507 (1972), 1–17.

[MR2] O. MARTIO and S. RICKMAN: *Measure properties of the branch set and its image of quasiregular mappings.* – Ann. Acad. Sci. Fenn. Ser. A I 541 (1973), 1–16.

[MRV1] O. MARTIO, S. RICKMAN, and J. VÄISÄLÄ: *Definitions for quasiregular mappings.* – Ann. Acad. Sci. Fenn. Ser. A I 448 (1969), 1–40.

[MRV2] O. MARTIO, S. RICKMAN, and J. VÄISÄLÄ: *Distortion and singularities of quasiregular mappings.* – Ann. Acad. Sci. Fenn. Ser. A I 465 (1970), 1–13.

[MRV3] O. MARTIO, S. RICKMAN, and J. VÄISÄLÄ: *Topological and metric properties of quasiregular mappings.* – Ann. Acad. Sci. Fenn. Ser. A I 488 (1971), 1–31.

[MS1] O. MARTIO and J. SARVAS: *Density conditions in the n–capacity.* – Indiana Univ. Math. J. 26 (1977), 761–776.

[MS2] O. MARTIO and J. SARVAS: *Injectivity theorems in plane and space.* – Ann. Acad. Sci. Fenn. Ser. A I Math. 4 (1978/79), 383–401.

[MSR1] O. MARTIO and U. SREBRO: *Periodic quasimeromorphic mappings.* – J. Analyse Math. 28 (1975), 20–40.

[MSR2] O. MARTIO and U. SREBRO: *Automorphic quasimeromorphic mappings in R^n.* – Acta Math. 135 (1975), 221–247.

[MSR3] O. MARTIO and U. SREBRO: *On the existence of automorphic quasimeromorphic mappings in R^n.* – Ann. Acad. Sci. Fenn. Ser. A I Math. 3 (1977), 123–130.

[MSR4] O. MARTIO and U. SREBRO: *Universal radius of injectivity for locally quasiconformal mappings.* – Israel J. Math. 29 (1978), 17–23.

[MV] O. MARTIO and M. VUORINEN: *Whitney cubes, p–capacity, and Minkowski content.* – Exposition. Math. 5 (1987), 17–40.

[MAT1] P. MATTILA: *Integration in a space of measures.* – Ann. Acad. Sci. Fenn. Ser. A I 555 (1973), 1–37.

[MAT2] P. MATTILA: *Lecture notes on geometric measure theory.* – Universidad de Extremadura, 1986.

[MATR] P. MATTILA and S. RICKMAN: *Averages of the counting function of a quasiregular mapping.* – Acta Math. 143 (1979), 273–305.

[MAZ1] V.G. MAZ'YA: *On the continuity at a boundary point of the solution of quasilinear elliptic equations.* (Russian). – Vestnik Leningrad. Univ. 25 (1970), No 13, 42–55.

[MAZ2] V.G. MAZ'YA: *Sobolev spaces.* (Russian). – Leningrad, 1985 (English translation Springer, 1986).

[MK] V.G. MAZ'YA and V.P. KHAVIN: *Non-linear potential theory.* (Russian). – Uspehi Mat. Nauk 27 (1972), 67–138.

[MIĆ] V.P. MIĆIĆ: *A theorem of Lindelöf type for quasiconformal mappings in space.* – Mat. Vesnik 9, 24 (1972), 3–8.

[MIK1] V.M. MIKLYUKOV: *On removable singularities of quasiconformal mappings in space.* (Russian). – Dokl. Akad. Nauk SSSR 188 (1969), 525–527.

[MIK2] V.M. MIKLYUKOV: *On a boundary property of n–dimensional mappings of bounded distortion.* (Russian). – Mat. Zametki 11 (1972), 159–164.

[MIK3] V.M. MIKLYUKOV: *Asymptotic properties of subsolutions of quasilinear equations of elliptic type and mappings with bounded distortion.* (Russian). – Mat. Sb. 11 (1980), 42–66.

[MIK4] V.M. MIKLYUKOV: *Some peculiarities of the behavior of solutions of minimal surface type equations in unbounded domains.* (Russian). – Mat. Sb. 116 (1981), 72–86.

[MI1] R. MINIOWITZ: *Distortion theorems for quasiregular mappings.* – Ann. Acad. Sci. Fenn. Ser. A I Math. 4 (1978/1979), 63–74.

[MI2] R. MINIOWITZ: *Spherically mean p–valent quasiregular mappings.* – Israel J. Math. 38 (1981), 199–208.

[MI3] R. MINIOWITZ: *Normal families of quasimeromorphic mappings.* – Proc. Amer. Math. Soc. 84 (1982), 35–43.

[MOR1] A. MORI: *On an absolute constant in the theory of quasiconformal mappings.* – J. Math. Soc. Japan 8 (1956), 156–166.

[MOR2] A. MORI: *On quasi–conformality and pseudo–analyticity.* – Trans. Amer. Math. Soc. 84 (1957), 56–77.

[MOS] J. MOSER: *On Harnack's theorem for elliptic differential equations.* – Comm. Pure and Appl. Math. 14 (1961), 577–591.

[MO] G.D. MOSTOW: *Quasiconformal mappings in n–space and the rigidity of hyperbolic space forms.* – Inst. Hautes Études Sci. Publ. Math. 34 (1968), 53–104.

[N1] R. NÄKKI: *Boundary behavior of quasiconformal mappings in n–space.* – Ann. Acad. Sci. Fenn. Ser. A I 484 (1970), 1–50.

[N2] R. NÄKKI: *Extension of Loewner's capacity theorem.* – Trans. Amer. Math. Soc. 180 (1973), 229–236.

[N3] R. NÄKKI: *Prime ends and quasiconformal mappings.* – J. Analyse Math. 35 (1979), 13–40.

[NP] R. NÄKKI and B. PALKA: *Lipschitz conditions and quasiconformal mappings.* – Indiana Univ. Math. J. 31 (1982), 377–401.

[NE] R. NEVANLINNA: *Analytic functions.* – Die Grundlehren der math. Wissenschaften Vol. 162, Springer-Verlag, Berlin–Heidelberg–New York, 1970.

[NO] K. NOSHIRO: *Cluster sets.* – Ergebnisse der Mathematik Vol. 28, Springer-Verlag, Berlin–Göttingen–Heidelberg, 1960.

[O] M. OHTSUKA: *Dirichlet problem, extremal length, and prime ends.* – Van Nostrand, Reinhold, 1970.

[ØK1] B. ØKSENDAL: *Stochastic methods in function theory.* – Proc. 19th Nordic Congr. Math. (Reykjavik, 1984), 255–270, Icelandic Math. Soc., Reykjavik, 1985.

[ØK2] B. ØKSENDAL: *Dirichlet forms, quasiregular functions and Brownian motion.* – Invent. Math. 91 (1988), 273–297.

[PA] P. PANSU: *Quasiconformal mappings and manifolds of negative curvature.* – Curvature and topology of Riemann manifolds. Proc. 17th Intern. Taniguchi symposium held in Katata, Japan, Aug. 26–31, 1985, Lecture Notes in Math. Vol. 1201, Springer-Verlag, Berlin–Heidelberg–New York, 1986.

[PE1] M. PESONEN: *A path family approach to Ahlfors's value distribution theory.* – Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 39 (1982), 1–32.

[PE2] M. PESONEN: *Simplified proofs of some basic theorems for quasiregular mappings.* – Ann. Acad. Sci. Fenn. Ser. A I Math. 8 (1983), 247–250.

[P1] E.A. POLETSKIĬ: *The modulus method for non–homeomorphic quasiconformal mappings.* (Russian). – Mat. Sb. 83 (1970), 261–272.

[P2] E.A. POLETSKIĬ: *On the removal of singularities of quasiconformal mappings.* (Russian). – Mat. Sb. 92 (134) (1973), 261–273.

[PS] G. PÓLYA and G. SZEGÖ: *Isoperimetric inequalities in mathematical physics.* – Princeton, Princeton University Press, 1951.

[PO] CH. POMMERENKE: *Univalent functions.* – Vandenhoeck & Ruprecht, Göttingen, 1975.

[PFTV] W.H. PRESS, B.P. FLANNERY, S.A. TEUKOLSKY, and W.T. VETTERLING: *Numerical recipies.* – Cambridge University Press, 1986.

[Q] H. QU: *An improvement of Mori's constant in the theory of quasiconformal mappings.* (Chinese). – J. Tongji Univ. 3 (1985), 75–85.

[RR] T. RADO and P.V. REICHELDERFER: *Continuous transformations in analysis.* – Die Grundlehren der math. Wissenschaften Vol. 75, Springer-Verlag, Berlin–Göttingen–Heidelberg, 1955.

[RE] E.G. REES: *Notes on geometry.* – Springer-Verlag, Berlin–Heidelberg–New York, 1983.

[REIR] H.M. REIMANN and T. RYCHENER: *Funktionen mit beschränkter mittlerer Oszillation.* – Lecture Notes in Math. Vol. 487, Springer-Verlag, Berlin–Heidelberg–New York, 1975.

[REN] H. RENGGLI: *Extremallängen und eine konform invariante Massfunktion für Kurvenscharen.* – Comment. Math. Helv. 41 (1966-67), 10–17.

[R1] YU. G. RESHETNYAK: *Estimates of the modulus of continuity for certain mappings.* (Russian). – Sibirsk. Mat. Zh. 7 (1966), 1106–1114.

[R2] YU.G. RESHETNYAK: *Space mappings with bounded distortion.* (Russian). – Sibirsk. Mat. Zh. 8 (1967), 629–659.

[R3] YU.G. RESHETNYAK: *The Liouville theorem with minimal regularity conditions.* (Russian). – Sibirsk. Mat. Zh. 8 (1967), 835–840.

[R4] YU. G. RESHETNYAK: *On the condition of the boundedness of index for mappings with bounded distortion.* (Russian). – Sibirsk. Mat. Zh. 9 (1968), 368–374.

[R5] YU. G. RESHETNYAK: *Mappings with bounded distortion as extremals of Dirichlet type integrals.* (Russian). – Sibirsk. Mat. Zh. 9 (1968), 652–666.

[R6] YU.G. RESHETNYAK: *Stability theorems for mappings with bounded distortion.* (Russian). – Sibirsk. Mat. Zh. 9 (1968), 667–684.

[R7] YU. G. RESHETNYAK: *The concept of capacity in the theory of functions with bounded distortion.* (Russian). – Sibirsk. Mat. Zh. 10 (1969), 1109–1138.

[R8] YU.G. RESHETNYAK: *On extremal properties of mappings with bounded distortion.* (Russian). – Sibirsk. Mat. Zh. 10 (1969), 1300–1310.

[R9] YU.G. RESHETNYAK: *The local structure of mappings with bounded distortion.* (Russian). - Sibirsk. Mat. Zh. 10 (1969), 1311-1313.

[R10] YU.G. RESHETNYAK: *On the branch set of mappings with bounded distortion.* (Russian). - Sibirsk. Mat. Zh. 11 (1970), 1333-1339.

[R11] YU.G. RESHETNYAK: *Stability in Liouville's theorem on conformal mappings of space for domains with a non–smooth boundary.* (Russian). – Sibirsk. Mat. Zh. 17 (1976), 361–369.

[R12] YU. G. RESHETNYAK: *Spatial mappings with bounded distortion.* (Russian). – Izdat. "Nauka", Sibirsk. Otdelenie, Novosibirsk, 1982.

[R13] YU. G. RESHETNYAK: *Stability theorems in geometry and analysis.* (Russian). – Izdat. "Nauka", Sibirsk. Otdelenie, Novosibirsk, 1982.

[RI1] S. RICKMAN: *Path lifting for discrete open mappings.* – Duke Math. J. 40 (1973), 187–191.

[RI2] S. RICKMAN: *A path lifting construction for discrete open mappings with application to quasimeromorphic mappings.* – Duke Math. J. 42 (1975), 797–809.

[RI3] S. RICKMAN: *On the value distribution of quasimeromorphic maps.* – Ann. Acad. Sci. Fenn. Ser. A I Math. 2 (1976), 447–466.

[RI4] S. RICKMAN: *On the number of omitted values of entire quasiregular mappings.* – J. Analyse Math. 37 (1980), 100–117.

[RI5] S. RICKMAN: *Asymptotic values and angular limits of quasiregular mappings of a ball.* – Ann. Acad. Sci. Fenn. Ser. A I Math. 5 (1980), 185–196.

[RI6] S. RICKMAN: *Value distribution of quasiregular mappings.* – 18th Scandinavian Congress of Mathematicians (Aarhus, 1980), Progr. Math., Vol. 11, 430–440, Birkhäuser, Boston, Mass., 1981.

[RI7] S. RICKMAN: *A defect relation for quasimeromorphic mappings.* – Ann. of Math. 114 (1981), 165–191.

[RI8] S. RICKMAN: *Value distribution of quasimeromorphic mappings.* – Ann. Acad. Sci. Fenn. Ser. A I Math. 7 (1982), 81–88.

[RI9] S. RICKMAN: *Value distribution of quasiregular mappings.* – Proc. Value Distribution Theory, Joensuu 1981, Lecture Notes in Math. Vol. 981, 220–245, Springer-Verlag, Berlin–Heidelberg–New York, 1983.

[RI10] S. RICKMAN: *Quasiregular mappings and metrics on the n-sphere with punctures.* – Comment. Math. Helv. 59 (1984), 134–148.

[RI11] S. RICKMAN: *The analogue of Picard's theorem for quasiregular mappings in dimension three.* – Acta Math. 154 (1985), 195–242.

[RI12] S. RICKMAN: *Book manuscript.* – In preparation.

[RV] S. RICKMAN and M. VUORINEN: *On the order of quasiregular mappings.* – Ann. Acad. Sci. Fenn. Ser. A I Math. 7 (1982), 221–231.

[RO] B. RODIN: *The method of extremal length.* – Bull. Amer. Math. Soc. 80 (1974), 587–606.

[S1] J. SARVAS: *Symmetrization of condensers in n-space.* – Ann. Acad. Sci. Fenn. Ser. A I 522 (1972), 1–44.

[S2] J. SARVAS: *The Hausdorff dimension of the branch set of a quasiregular mapping.* – Ann. Acad. Sci. Fenn. Ser. A I Math. 1 (1975), 297–307.

[S3] J. SARVAS: *Coefficient of injectivity for quasiregular mappings.* – Duke Math. J. 43 (1976), 147–158.

[SC] H. SCHWERDTFEGER: *Geometry of complex numbers.* – Dover Publ., New York, 1979.

[SEM1] V. I. SEMENOV: *An integral representation of the trace on the sphere of a class of vector fields and uniform stability estimates of quasiconformal mappings of the ball.* (Russian). – Mat. Sb. 133 (1987), 238–253.

[SEM2] V. I. SEMENOV: *On some dynamic systems and quasiconformal mappings.* (Russian). – Sibirsk. Mat. Zh. 28 (1987), 196–206.

[SE] J. SERRIN: *Local behavior of solutions of quasilinear equations.* – Acta Math. 111 (1964), 247–302.

[SH] B.V. SHABAT: *On the theory of quasiconformal mappings in space.* – Soviet Math. 1 (1960), 730–733.

[SO] S.L. SOBOLEV: *Applications of functional analysis in mathematical physics.* – Amer. Math. Soc., Rhode Island, 1963.

[SR1] U. SREBRO: *Conformal capacity and quasiconformal mappings in R^n.* – Israel J. Math. 9 (1971), 93–110.

[SR2] U. SREBRO: *Conformal capacity and quasiregular mappings.* – Ann. Acad. Sci. Fenn. Ser. A I 529 (1973), 1–8.

[SR3] U. SREBRO: *Quasiregular mappings.* – In: Advances in Complex Analysis pp. 148–163, Lecture Notes in Math. Vol. 505, Springer-Verlag, Berlin–Heidelberg–New York, 1976.

[ST] E. M. STEIN: *Singular integrals and differentiability properties of functions.* – Princeton Univ. Press, Princeton, New Jersey, 1970.

[STR1] E.D. STREDULINSKY: *Higher integrability from reverse Hölder inequalities.* – Indiana Univ. Math. J. 29 (1980), 407–413.

[STR2] E.D. STREDULINSKY: *Weighted inequalities and degenerate elliptic partial differential equations.* – Lecture Notes in Math. Vol. 1074, Springer-Verlag, Berlin–Heidelberg–New York, 1984.

[SU] G.D. SUVOROV: *Generalized length-area principle in the theory of mappings.* (Russian). – Kiev, Naukova Dumka, 1985.

[SY] A.V. SYCHEV: *Moduli and n-dimensional quasiconformal mappings.* (Russian). – Izdat. "Nauka", Sibirsk. Otdelenie, Novosibirsk, 1983.

[TE] O. TEICHMÜLLER: *Untersuchungen über konforme und quasikonforme Abbildung.* – Deutsche Math. 3 (1938), 621–678.

[TY] C. J. TITUS and G. S. YOUNG: *The extension of interiority, with some applications.* – Trans. Amer. Math. Soc. 103 (1962), 329–340.

[TR] D. A. TROTSENKO: *Approximation by similarities of mappings with bounded distortion.* (Russian). – Sibirsk. Mat. Zh. 27 (1986), 198–205.

[T] M. TSUJI: *Potential theory in modern function theory.* – Second ed., Chelsea Publishing Co, New York, 1975.

[TU1] P. TUKIA: *A quasiconformal group not isomorphic to a Möbius group.* – Ann. Acad. Sci. Fenn. Ser. A I Math. 6 (1981), 149–160.

[TU2] P. TUKIA: *On quasiconformal groups.* – J. Analyse Math. 46 (1986), 318–345.

[TV] P. TUKIA and J. VÄISÄLÄ: *Quasisymmetric embeddings of metric spaces.* – Ann. Acad. Sci. Fenn. Ser. A I Math. 5 (1980), 97–114.

[V1] J. VÄISÄLÄ: *On quasiconformal mappings in space.* – Ann. Acad. Sci. Fenn. Ser. A I 298 (1961), 1–36.

[V2] J. VÄISÄLÄ: *On quasiconformal mappings of a ball.* – Ann. Acad. Sci. Fenn. Ser. A I 304 (1961), 1–17.

[V3] J. VÄISÄLÄ: *On the null-sets for extremal lengths.* – Ann. Acad. Sci. Fenn. Ser. A I 322 (1962), 1–12.

[V4] J. VÄISÄLÄ: *Minimal mappings in euclidean spaces.* – Ann. Acad. Sci. Fenn. Ser. A I 366 (1965), 1–22.

[V5] J. VÄISÄLÄ: *Discrete open mappings on manifolds.* – Ann. Acad. Sci. Fenn. Ser. A I 392 (1966), 1–10.

[V6] J. VÄISÄLÄ: *Removable sets for quasiconformal mappings.* – J. Math. Mech. 19 (1969), 49–51.

[V7] J. VÄISÄLÄ: *Lectures on n-dimensional quasiconformal mappings.* – Lecture Notes in Math. Vol. 229, Springer-Verlag, Berlin–Heidelberg–New York, 1971.

[V8] J. VÄISÄLÄ: *Modulus and capacity inequalities for quasiregular mappings.* – Ann. Acad. Sci. Fenn. Ser. A I 509 (1972), 1–14.

[V9] J. VÄISÄLÄ: *Capacity and measure.* – Michigan Math. J. 22 (1975), 1–3.

[V10] J. VÄISÄLÄ: *A survey of quasiregular maps in R^n.* – Proc. Internat. Congr. Math. (Helsinki, 1978), Vol. 2, 685–691, Acad. Sci. Fennica, Helsinki, 1980.

[V11] J. VÄISÄLÄ: *Quasisymmetric embeddings in euclidean spaces.* – Trans. Amer. Math. Soc. 264 (1981), 191–204.

[V12] J. VÄISÄLÄ: *Quasimöbius maps.* – J. Analyse Math. 44 (1984/85), 218–234.

[V13] J. VÄISÄLÄ: *The degree of a mapping.* – Seminar on quasiregular mappings 1968–1969, Helsinki (unpublished lecture notes in Finnish).

[V14] J. VÄISÄLÄ: *Quasiregular mappings in n-space.* – XVI Scand. Congr. of Math. (Gothenburg, 1972) (unpublished).

[VGR] S. K. VODOP'YANOV, V. M. GOL'DSTEIN, and YU. G. RESHETNYAK: *On geometric properties of functions with generalized first derivatives.* (Russian). – Uspehi Mat. Nauk 34 (1979), 17–62.

[VU1] M. VUORINEN: *Exceptional sets and boundary behavior of quasiregular mappings in n-space.* – Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 11 (1976), 1–44.

[VU2] M. VUORINEN: *Lower bounds for the moduli of path families with application to non-tangential limits of quasiconformal mappings.* – Ann. Acad. Sci. Fenn. Ser. A I Math. 4 (1978/1979), 279–291.

[VU3] M. VUORINEN: *On the existence of angular limits of n-dimensional quasiconformal mappings.* – Ark. Mat. 18 (1980), 157–180.

[VU4] M. VUORINEN: *On the boundary behavior of locally K-quasiconformal mappings in space.* – Ann. Acad. Sci. Fenn. Ser. A I Math. 5 (1980), 79–95.

[VU5] M. VUORINEN: *Capacity densities and angular limits of quasiregular mappings.* – Trans. Amer. Math. Soc. 263 (1981), 2, 343–354.

[VU6] M. VUORINEN: *On the Harnack constant and the boundary behavior of Harnack functions.* – Ann. Acad. Sci. Fenn. Ser. A I Math. 7 (1982), 259–277.

[VU7] M. VUORINEN: *Lindelöf-type theorems for quasiconformal and quasiregular mappings.* – Proc. Complex Analysis Semester, Banach Center Public., Vol. 11, 353–362, Warsaw, 1983.

[VU8] M. VUORINEN: *Some inequalities for the moduli of curve families.* – Michigan Math. J. 30 (1983), 369–380.

[VU9] M. VUORINEN: *On functions with a finite or locally bounded Dirichlet integral.* – Ann. Acad. Sci. Fenn. Ser. A I Math. 9 (1984), 177–194.

[VU10] M. VUORINEN: *Conformal invariants and quasiregular mappings.* – J. Analyse Math. 45 (1985), 69–115.

[VU11] M. VUORINEN: *On the distortion of n-dimensional quasiconformal mappings.* – Proc. Amer. Math. Soc. 96 (1986), 275–283.

[VU12] M. VUORINEN: *On quasiregular mappings and domains with a complete conformal metric.* – Math. Z. 194 (1987), 459–470.

[VU13] M. VUORINEN: *On Teichmüller's modulus problem in R^n.* – Math. Scand. (to appear).

[VU14] M. VUORINEN: *On Picard's theorem for quasiregular mappings.* – In preparation.

[W1] H. WALLIN: *A connection between α-capacity and L^p-classes of differentiable functions.* – Ark. Mat. 5 (1964), 331–341.

[W2] H. WALLIN: *Metrical characterization of conformal capacity zero.* – J. Math. Anal. Appl. 58 (1977), 298–311.

[WA] C.-F. WANG: *On the precision of Mori's theorem in Q-mapping.* – Science Record 4 (1960), 329–333.

[WH1] G.T. WHYBURN: *An open mapping approach to Hurwitz's theorem.* – Trans. Amer. Math. Soc. 71 (1951), 113–119.

[WH2] G. T. WHYBURN: *Topological analysis.* – Princeton University Press, Second ed. 1964, Princeton.

[WI] J.B. WILKER: *Inversive geometry.* – The geometric vein. The Coxeter Festschrift, 379–442, Springer-Verlag, New York–Heidelberg–Berlin, 1982.

[Z1] W.P. ZIEMER: *Extremal length and p-capacity.* – Michigan Math. J. 16 (1969), 43–51.

[Z2] W.P. ZIEMER: *Extremal length as a capacity.* – Michigan Math. J. 17 (1970), 117–128.

[ZI] M. ZINSMEISTER: *Domaines des Lavrentiev.* – Publications mathematiques d'Orsay (1985).

[ZO1] V. A. ZORICH: *The theorem of M. A. Lavrent'ev on quasiconformal mappings in space.* (Russian). – Mat. Sb. 74 (1967), 417–433.

[ZO2] V. A. ZORICH: *Isolated singularities of mappings with bounded distortion.* (Russian). – Mat. Sb. 81 (1970), 634–638.

Index

A list of special symbols is given on pages XVII–XIX.